$q-$Pascal's triangle and irreducible representations of the braid group B_3 in arbitrary dimension

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Abstract

We construct a $\left[\frac{n+1}{2}\right]+1$ parameters family of irreducible representations of the Braid group B_3 in arbitrary dimension $n\in\mathbb{N}$, using a $q-$deformation of the Pascal triangle. This construction extends in particular results by S.P. Humphries [8], who constructed representations of the braid group B_3 in arbitrary dimension using the classical Pascal triangle. E. Ferrand [7] obtained an equivalent representation of B_3 by considering two special operators in the space $\mathbb{C}^n[X]$. Slightly more general representations were given by I. Tuba and H. Wenzl [11]. They involve $[\frac{n+1}{2}]$ parameters (and also use the classical Pascal triangle). The latter authors also gave the complete classification of all simple representations of B_3 for dimension $n\leq 5$. Our construction generalize all mentioned results and throws a new light on some of them. We also study the irreducibility and the equivalence of the representations.

In [17] we establish the connection between the constructed representation of the braid group B_3 and the highest weight modules of $U(\mathfrak{s}l_2)$ and quantum group $U_q(\mathfrak{s}l_2)$.

Key words: Braid group, $\mathrm{SL}(2,\mathbb{Z})$, representations, classification, Pascal's triangle, $q-$Pascal's triangle, $q-$binomial coefficient, Gaussian polynomials, quantum groups
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Contents

1 Introduction

Let B_3 be Artin's braid group, given by the generators σ_1 and σ_2 and the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ [5]. Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{N} := \{0, 1, 2, ...\}$ and let $\mathrm{Mat}(n, \mathbb{C})$ be the set of complex $n \times n$ matrices. We construct irreducible representations of the braid group B_3 in the space $\mathbb{C}^{n+1}$

for an arbitrary $n \in \mathbb{N}$ using the $q-$Pascal triangle, i.e. the $q-$deformation of the usual Pascal triangle.

Let $\binom{n}{k}_q$, $k, n \in \mathbb{N}$ be the $q-$binomial coefficients (see the definition in Section 2). For the matrix $A = (a_{km})_{0\leq k,m,\leq n} \in \mathrm{Mat}(n+1,\mathbb{C})$ we set $A^\sharp = (a^\sharp_{ij})$ where $a^\sharp_{ij} = a_{n-i,n-j}$. In all sections except Section 9 we index row and columns of the matrix $A \in \mathrm{Mat}(n+1,\mathbb{C})$ starting from 0. For any nonzero $q \in \mathbb{C}$ let the matrix $\Lambda_n(q)$ and the numbers q_n be defined as follows:

$$\Lambda_n(q) = \mathrm{diag}\,(q_{rn})_{r=0}^n\,, \text{ where } q_{rn} := \frac{q_r q_{n-r}}{q_n} = q^{-(n-r)r} \text{ and } q_n = q^{\frac{(n-1)n}{2}},\ n \in \mathbb{N}. \tag{1}$$

For an arbitrary $n \in \mathbb{N}$ and a complex diagonal matrix $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_n)$ we define our **representation of the group B_3 in the space $\mathbb{C}^{n+1}$** by following formulas

$$\sigma_1 \mapsto \sigma_1^\Lambda := \sigma_1(q,n)\Lambda \quad \text{ and } \sigma_2 \mapsto \sigma_2^\Lambda := \Lambda^\sharp\sigma_2(q,n) = \Lambda^\sharp(\sigma_1^{-1}(q^{-1},n))^\sharp, \tag{2}$$

where $\sigma_1(q,n) = (\sigma_1(q,n)_{km})_{0\leq k,m\leq n}$ and $\sigma_2(q,n)$ are defined by

$$\sigma_1(q,n)_{km} = \sigma_1(q)_{km} = \binom{n-k}{n-m}_q,\ 0 \leq k,m \leq n,\ \ \sigma_2(q,n) = (\sigma_1^{-1}(q^{-1},n))^\sharp, \tag{3}$$

(as usually, we set $\binom{n}{k}_q = 0$ for $k > n$), and the matrix $\Lambda = \Lambda_n$ satisfies the following condition:

$$\lambda_0\lambda_n\Lambda_n(q) = \Lambda_n\Lambda_n^\sharp \quad \text{or} \quad \lambda_0\lambda_n\frac{q_r q_{n-r}}{q_n} = \lambda_r\lambda_{n-r},\ 0 \leq r \leq n. \tag{4}$$

The aim of this article is to show that formulas (2) give an $\left[\frac{n+1}{2}\right]+1$ parameters family of B_3 representations in dimension $n+1$, for any $n \in \mathbb{N}$ (Theorem 1) and study the irreducibility (Theorem 3,4) and the equivalence (Theorem 5).

In Section 2 we introduce the main objects and give the main statements. In Section 3 we present the result of S.P. Humphries [11] who has constructed a representation of B_3 equivalent with the particular case of our representation when $q = 1$ and $\Lambda = I$

$$\sigma_1 \mapsto \sigma_1(1,n), \quad \sigma_2 \mapsto \sigma_2(1,n). \tag{5}$$

In his representations the classical Pascal triangle plays a basic role. In Section 4 we mention the result of E. Ferrand [8] who has constructed a representation of B_3 equivalent with the representation (5). For this E. Ferrand has considered two operators $\Phi : p(x) \mapsto p(x+1)$ and $\Psi : p(x) \mapsto (1-x)^n p(\frac{x}{1-x})$ in the space $\mathbb{C}^n[X]$ of the polynomials of degree $\leq n$ satisfying the relation $\Phi\Psi\Phi = \Psi\Phi\Psi$. The $q-$analogue of the mentioned results is given in Section 13.

In Section 5 we show that this representation is closely connected with the morphism $\rho : B_3 \mapsto \mathrm{SL}(2,\mathbb{Z})$ (see (33) below) and the nth symmetric power of the natural representation $\pi : \mathrm{SL}(2,\mathbb{Z}) \mapsto \mathrm{SL}(2,\mathbb{Z})$. Section 6 recalls shortly the results of I. Tuba and H. Wenzl [19]. Firstly, they showed that

$$\sigma_1 \mapsto \sigma_1(1,n)\Lambda, \quad \sigma_2 \mapsto \Lambda^\sharp \sigma_2(1,n) \tag{6}$$

is a representation of B_3 in arbitrary dimension $n+1$ if $\lambda_r \lambda_{n-r} = c$ for some constant c (see Remark 6.2, Section 2). Secondly, they gave the complete classification of all irreducible representations of B_3 for dimension ≤ 5.

Our motivation for the present study was to generalize the results and formulas of the mentioned authors to the case of an arbitrary dimension $n \in \mathbb{N}$. We have realized that not only the classical Pascal triangle may be used for constructing of the representations of B_3 but also the $q-$deformed Pascal triangle. The conditions $\lambda_r \lambda_{n-r} = c$ on Λ in the classical case should be replaced in the deformed case by some rather nontrivial conditions (see (4)) connecting the matrix Λ with some canonical diagonal matrix $\Lambda_n(q)$, depending on q and n. We prove that the representations of B_3 given by (2) coincide with the representations of I. Tuba and H. Wenzl [19] for $n = 4$, are equivalent with them for $n = 2, 3, 5$, and generalize them for an arbitrary dimension n (Remark 6.3, Section 2). This is explained in Section 10.

In Section 7 we present the results of S.P. Humphries and the results of I. Tuba and H. Wenzl in a form which is convenient for our extensions. In Section 8 we show how the Pascal (resp. $q-$Pascal) triangle appears as the operators $\exp T_1$ (resp. $\exp_{(q)} T_{(q)}$) associated with some operators T_1 (resp. $T_{(q)}$). The irreducibility and the equivalence of our representations (Theorem 3, 4 and 5) are studied in Section 9.

In Section 11 we give the proof of the Theorem 1 i.e. that (2) is a representation. In Section 12 we prove some combinatorial identities for $q-$binomial coefficients. These identities are an essential part in the proof of the Theorem 1. They generalize the well-known combinatorial identities for classical binomial coefficients (see [10]) used by S.P. Humphries to prove that (5) is a representation of B_3.

Let us also mention that in the article of S. Albeverio and S. Rabanovich [2] a class of unitary irreducible representations of B_3 by $n \times n$ matrices for every $n \geq 3$ was constructed. Using tensor products of these representations and the reduced Barrau representations [12] these authors also find a class of irreducible unitary representations of B_4.

In [9] E.Formanek et al. gave the *complete classification* of all *simple representations of B_n* for *dimension* $\leq n$. To know more on the braid groups and its applications see [6,7].

2 Main objects

The Pascal triangle consists of the binomial coefficients $\binom{n}{k} := C_n^k,\ k, n \in \mathbb{N}$, defined by

$$C_n^k := \frac{n!}{k!(n-k)!},\ 0 \le k \le n, C_n^k = 0,\ k > n,\ \text{where } n! = 1 \cdot 2 \cdot ... \cdot n. \quad (7)$$

We recall that the binomial coefficients may also be defined by induction, using the relations

$$C_n^0 = C_n^n = 1,\ n \in \mathbb{N}, \quad C_{n+1}^k = C_n^{k-1} + C_n^k, \quad 1 \le k \le n.$$

We also consider the $q-$Pascal triangle consisting of $q-$binomial coefficients $\binom{n}{k}_q = C_n^k(q),\ 0 \le k \le n,\ n \in \mathbb{N}$ defined as follows (see [3,13,14,15])

$$\binom{n}{k}_q := C_n^k(q) = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},\ 0 \le k \le n, \quad C_n^k(q) = 0,\ k > n, \quad (8)$$

where $(a;q)_n$ denotes the standard $q-$shifted factorial [3,4,15]

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}). \quad (9)$$

The $q-$binomial coefficients were first studied by Gauss and have come to be known as *Gaussian polynomials* (see [3, Ch. 3.3]). Another definition [14, Ch.IV.2] of the Gauss polynomials is following. For any integer $n > 0$ we define the associated $q-$*integer* $(n)_q$ by

$$(n)_q = 1 + q + \dots + q^{n-1} = \frac{1-q^n}{1-q}. \quad (10)$$

Define the $q-$*factorial* of n by $(0)!_q = 1$ and

$$(n)!_q = (1)_q(2)_q \dots (n)_q = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}, \quad (11)$$

when $n > 0$. We define the Gaussian polynomials for $0 \le k \le n$ by

$$C_n^k(q) = \frac{(n)!_q}{(k)!_q(n-k)!_q}. \quad (12)$$

They can also be defined recursively, using the following relations ([14, Ch.IV.2], see also [13]) $C_n^0(q) = C_n^n(q) = 1,\ n \in \mathbb{N}:$

$$C_{n+1}^k(q) = C_n^{k-1}(q) + q^k C_n^k(q),\ C_{n+1}^k(q) = q^{n-k} C_n^{k-1}(q) + C_n^k(q),\ 1 \le k \le n. \quad (13)$$

We can also obtain the Gaussian polynomials in the following way. Let us set

$$(1+x)_q^k := (1+x)(1+xq)(1+xq^2)\dots(1+xq^{n-1}).$$

We have (see [3])

$$(1+x)_q^k = \sum_{r=0}^{k} q^{r(r-1)/2} C_k^r(q) x^r = \sum_{r=0}^{k} q^{r(r-1)/2} \binom{k}{r}_q x^r. \tag{14}$$

As an example we make explicit the corresponding $q-$Pascal triangle for $n=5$:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 1+q & & 1 & & & \\
 & & 1 & & 1+q+q^2 & & C_3^2(q) & & 1 & & \\
 & 1 & & (1+q)(1+q^2) & & (1+q^2)(1+q+q^2) & & C_4^3(q) & & 1 & \\
1 & & 1+q+q^2+q^3+q^4 & & (1+q^2)(1+q+q^2+q^3+q^4) & & C_5^3(q) & & C_5^4(q) & & 1
\end{array}$$

Notations. For an $n\times n$ matrix $A=(a_{ij})$ we set A^t (resp. A^s and $A^\sharp$) where

$$A^t=(a_{ij}^t),\ a_{ij}^t=a_{ji},\ (\text{resp } A^s=(a_{ij}^s),\ \ a_{ij}^s=a_{n-j,n-i};\ \ A^\sharp=(a_{ij}^\sharp),\ \ a_{ij}^\sharp=a_{n-i,n-j}). \tag{15}$$

The operation $A\to A^\sharp$ means composing the transposition with respect to the main diagonal ($A\to A^t$) with the transposition with respect to the auxiliary (subsidiary) diagonal ($A\to A^s$) i.e. $A^\sharp=(A^t)^s=(A^s)^t$.

Let us consider the $(n+1)\times(n+1)$ matrix $S(q)$ defined as follows:

$$S(q)=(S(q)_{km}),\qquad \text{where } S(q)_{km}=q_k^{-1}(-1)^k\delta_{k+m,n},\ S:=S(1), \tag{16}$$

$$S(q)=\begin{pmatrix}
0 & \dots & 0 & 0 & 0 & 1\\
0 & \dots & 0 & 0 & -1 & 0\\
0 & \dots & 0 & q^{-1} & 0 & 0\\
0 & \dots & -q^{-3} & 0 & 0 & 0\\
 & \dots & & & & \\
(-1)^n q^{-\frac{(n-1)n}{2}} & \dots & 0 & 0 & 0 & 0
\end{pmatrix} \ \text{and} \ \ S=\begin{pmatrix}
0 & \dots & 0 & 0 & 0 & 1\\
0 & \dots & 0 & 0 & -1 & 0\\
0 & \dots & 0 & 1 & 0 & 0\\
0 & \dots & -1 & 0 & 0 & 0\\
 & \dots & & & & \\
(-1)^n & \dots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Theorem 1 *Formulas (2) define a representation of B_3 in the space $\mathbb{C}^{n+1}$ for an arbitrary $n\in\mathbb{N}$ i.e.*

$$\sigma_1^\Lambda\sigma_2^\Lambda\sigma_1^\Lambda=\sigma_2^\Lambda\sigma_1^\Lambda\sigma_2^\Lambda=\lambda_0\lambda_n S(q)\Lambda, \tag{17}$$

moreover

$$(\sigma_1^{-1}(q^{-1}))_{km}^\sharp=\begin{cases} 0, & \text{if } 0<k<m\le n,\\ (-1)^{k+m}q_{k-m}^{-1}C_k^m(q^{-1}), & \text{if } 0\le m\le k\le n.\end{cases} \tag{18}$$

Remark 2 *1. Let us set*

$$D_n(q)=\operatorname{diag}(q_r)_{r=0}^n. \tag{19}$$

We have by (1) and (16)

$$\Lambda(q) = q_n^{-1} D_n(q) D_n^\sharp(q), \quad S(q) = D_n^{-1}(q) S, \tag{20}$$

so if we take $\Lambda = D_n(q)$ *or* $\Lambda = D_n^\sharp(q)$ *the relation (4) is satisfied, hence*

$$\sigma_1 \mapsto \sigma_1^D(q,n) := \sigma_1(q,n) D_n^\sharp(q) \quad \text{and} \quad \sigma_2 \mapsto \sigma_2^D(q,n) := D_n(q)\sigma_2(q,n) \tag{21}$$

also gives a representation of the braid group B_3.
2. The general form of the matrix Λ_n *satisfying (4) is following:* $\Lambda_n = D_n^\sharp(q)\Lambda_n'$ *or* $\Lambda_n = D_n(q)\Lambda_n'$ *where* $\Lambda_n' = \mathrm{diag}(\lambda_0', \lambda_1', ..., \lambda_n')$ *with* $\Lambda_n'(\Lambda_n')^\sharp = cI$ *for some constant* c.

Using Remark 2 we shall write our representation in the following form

$$\sigma_1 \mapsto \sigma_1^\Lambda(q,n) = \sigma_1(q,n) D_n^\sharp(q)\Lambda_n, \ \sigma_2 \mapsto \sigma_2^\Lambda(q,n) = \Lambda_n^\sharp D_n(q)\sigma_2(q,n), \ \Lambda_n\Lambda_n^\sharp = cI. \tag{22}$$

Definition. *We say that the representation is* **subspace irreducible** *or* **irreducible** *(resp.* **operator irreducible***) when there no nontrivial invariant close* **subspaces** *for all operators of the representation (resp. there no nontrivial bounded* **operators** *commuting with all operators of the representation).*

Let us define for n, r, q, λ such that $n \in \mathbb{N}, \ 0 \le r \le n, \ \lambda \in \mathbb{C}^{n+1}, \ q \in \mathbb{C}$ the following operators

$$F_{r,n}(q,\lambda) = \exp_{(q)}\left(\sum_{k=0}^{n-1}(k+1)_q E_{kk+1}\right) - q_{n-r}\lambda_r (D_n(q)\Lambda_n^\sharp)^{-1}, \tag{23}$$

where $\exp_{(q)} X = \sum_{m=0}^\infty X^m/(m)!_q$. For the matrix $C \in \mathrm{Mat}(n+1, \mathbb{C})$ we denote by

$$M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C), \ (\text{resp. } A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C)), \ 0 \le i_1 < ... < i_r \le n, \ 0 \le j_1 < ... < j_r \le n$$

its minors (resp. the cofactors) with $i_1, i_2, ..., i_r$ rows and $j_1, j_2, ..., j_r$ columns.

Theorem 3 *The representation of the group* B_3 *defined by (22) have the following properties:*
1) for $q = 1, \ \Lambda_n = 1$, *it is* subspace irreducible *in arbitrary dimension* $n \in \mathbb{N}$;
2) for $q \ne 1, \ \Lambda_n = \mathrm{diag}(\lambda_k)_{k=0}^n \ne 1$ *it is* operator irreducible *if and only if for any* $0 \le r \le \left[\frac{n}{2}\right]$ *there exists* $0 \le i_0 < i_i < ... < i_r \le n$ *such that*

$$M^{i_0 i_i \dots i_{n-r-1}}_{r+1 r+2 \dots n}(F^s_{r,n}(q,\lambda)) \ne 0; \tag{24}$$

3) for $q \ne 1, \ \Lambda_n = 1$ *it is subspace irreducible if and only if* $(n)_q \ne 0$.
The representation has $[\frac{n+1}{2}] + 1$ *free parameters.*

Let us denote by $\sigma^{\Lambda}(q,n)$ the representation of B_3 defined by (22).

Theorem 4 *The representation* $\sigma^{\Lambda}(q,n)$ *is* subspace irreducible *for* $n=1$ *if and only if* $\Lambda_1 \neq \lambda_0(1,\alpha)$ *where* $\alpha^2-\alpha+1=0$.

Problem. *To find a criteria of the subspace irreducibility for all representations* $\sigma^{\Lambda}(q,n)$*. Some particular cases are studied in Section 8.*

Theorem 5 *If two represenations* $\sigma^{\Lambda}(q,n)$ *and* $\sigma^{\Lambda'}(q',n)$ *are equivalent i.e.*

$$\sigma_i^{\Lambda}(q,n)C = C\sigma_i^{\Lambda'}(q',n),\ \ i=1,2$$

for some $C \in \mathrm{GL}(n+1,\mathbb{C})$ *then* $q/q'=1$ *for* $n=2m$ *and* $(q/q')^2=1$ *for* $n=2m-1$.

Remark 6 *1. In the particular case where* $\Lambda = I$ *and* $q=1$ *Theorem 1 gives the result of S.P. Humphries [11] (see Section 3).*
2. When $q=1$ *and* $\Lambda = \mathrm{diag}(\lambda_0,\lambda_1,...,\lambda_n)$ *we obtain the example of I. Tuba and H. Wenzl [19] (see Section 5, Example 1).*
3. The representations of B_3 *given by (22) coincide with the representations of I. Tuba and H. Wenzl [19] for* $n=4$*, are equivalent with them in the dimension* $n=2,3,5$*, and generalize them for an arbitrary dimension* n*.*
4. Using Theorem 3 and 4 we give in Section 9.5 examples of representations of B_3 *that are operator irreducible but are subspace reducible.*

Theorem 7 *In particular using result of [19] (Sections 2.4-2.7) we conclude that all irreducible representations of* B_3 *for dimension* ≤ 5 *are given by (22).*

3 Pascal's triangle and representations of B_3. Results of Humphries

Following S.P. Humphries [11], for fixed $n \geq 1$ we let $\Sigma_1 = \Sigma_1(n)$ and $\Sigma_2 = \Sigma_2(n)$ (respectively) be the following $(n+1)\times(n+1)$ lower and upper (respectively) triangular matrices:

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ & & & & & ... & \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & ... & \binom{n}{n} \end{pmatrix}, \quad \begin{pmatrix} \binom{n}{n} & ... & \binom{n}{4} & \binom{n}{3} & \binom{n}{2} & \binom{n}{1} & \binom{n}{0} \\ & & & & & ... & \\ & & 1 & 4 & 6 & 4 & 1 \\ & & & 1 & 3 & 3 & 1 \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{pmatrix},$$

(Thus we make the convention that a blank indicates the zero entry). Let $E = E_n$ be the $(n+1) \times (n+1)$ permutation matrix corresponding to the permutation $(0n)(1n-1)(2n-2)$.... S.P. Humphries shows that

$$\sigma_1 \mapsto \Sigma_1, \quad \sigma_2 \mapsto \Sigma_2^{-1} \tag{25}$$

gives a representation of B_3 using the following lemmas.

Lemma 4.1 *We have $E\Sigma_1 E^{-1} = \Sigma_2$. Further*

$$\Sigma_2^{-1} = \begin{pmatrix} \binom{n}{n} & ... & (-1)^{n-4}\binom{n}{4} & (-1)^{n-3}\binom{n}{3} & (-1)^{n-2}\binom{n}{2} & (-1)^{n-1}\binom{n}{1} & (-1)^{n}\binom{n}{0} \\ & ... & & & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -3 & 3 & -1 \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \\ & & & & & & 1 \end{pmatrix}$$

There is a similar expression for Σ_1^{-1}, namely $\Sigma_1^{-1} = E^{-1}\Sigma_2^{-1}E$.

Lemma 4.2 *We have*

$$\Sigma_1\Sigma_2^{-1} = \begin{pmatrix} 1 & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \binom{n}{4} & ... & 1 \\ & & & & & ... & \\ 1 & -4 & 6 & -4 & 1 & & \\ 1 & -3 & 3 & -1 & & & \\ 1 & -2 & 1 & & & & \\ 1 & -1 & & & & & \\ 1 & & & & & & \end{pmatrix}$$

Lemma 4.3 *We have $\Sigma_1\Sigma_2^{-1}\Sigma_1 = \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1} = (-1)^n G_n$, where G_n is the $(n+1) \times (n+1)$ matrix $\mathrm{diag}(1, -1, 1, ...)E_n$.*

4 Pascal's triangle in the space $\mathbb{C}^n[X]$ and results of E. Ferrand

In the work of E. Ferrand [8] the Pascal triangle appears in the following way. Denote by Φ the endomorphism of the space $\mathbb{C}^n[X]$ of polynomials of degree

n with complex coefficients, which maps a polynomial $p(X)$ to the polynomial $p(X+1)$. Denote by Ψ the endomorphism of $\mathbb{C}^n[X]$ which maps a a polynomial $p(X)$ to $(1-X)^n p\left(\frac{X}{1-X}\right)$

Theorem [8]. *Φ and Ψ verify a braid-like relation $\Phi\Psi\Phi = \Psi\Phi\Psi$.*

One can verify that Φ (resp. Ψ) in the canonical basis $1, X, X^2, ..., X^n$ of $\mathbb{C}^n[X]$ have the form

$$\Phi = \Sigma_2(n)^s = \sigma_1^s(1,n), \quad \Psi = (\Sigma_1(n)^{-1})^s = (\sigma_2^{-1}(1,n))^s, \tag{26}$$

where the notations A^s is defined in (15). For the operator Φ we have

$$X^k \overset{\Phi}{\mapsto} (1+X)^k = \sum_{r=0}^{k} C_k^r X^r \tag{27}$$

hence $\Phi_{rk} = C_k^r$ and we get the first part of (26) if we compare (27) with (3). For the operator Ψ we get

$$X^k \overset{\Psi}{\mapsto} (1-X)^{n-k} X^k = \sum_{r=0}^{n-k} (-1)^r C_{n-k}^r X^{r+k} = \sum_{t=k}^{n} (-1)^{k+t} C_{n-k}^{t-k} X^t \tag{28}$$

if we set $r + k = t$. Hence $\Psi_{tk} = (-1)^{k+t} C_{n-k}^{t-k}$ and we get the second part of (26) if we compare (28) with (18). Since $\Phi\Psi\Phi = \Psi\Phi\Psi$ we have another proof of the braid relations given in [11]: $\Sigma_1\Sigma_2^{-1}\Sigma_1 = \Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}$.

5 Pascal's triangle as the symmetric power

The representation of B_3 given by E. Ferrand can be obtained in the following way. There is a morphism $\rho : B_3 \mapsto \mathrm{SL}(2,\mathbb{Z})$ of the group B_3 in $\mathrm{SL}(2,\mathbb{Z})$ defined by (33) below. Let us consider the natural representation π of the group $\mathrm{SL}(2,\mathbb{Z})$ in the space $\mathbb{C}^1[X]$ defined as follows

$$(\pi_g f)(x) = (cx+d) f\left(\frac{ax+b}{cx+d}\right), \quad \text{where } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}).$$

We show that (see (3) for the notation $\sigma_1(1,n)$)

$$\mathrm{Sym}^n(\pi) \circ \rho(\sigma_k) = \sigma_k(1,n), \;\; k = 1, 2, \;\; n \in \mathbb{N}, \tag{29}$$

where $\mathrm{Sym}^n(\pi)$ is the symmetric power of the representation π. We have

$$\rho(\sigma_1) = \sigma_1(1,1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \rho(\sigma_2) = \sigma_2(1,1) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right).$$

Then (29) is transformed into

$$\mathrm{Sym}^n(\sigma_1(1,1)) = \sigma_1(1,n). \tag{30}$$

Let us take the basis e_0, e_1 of the space $V := \mathbb{C}^1[X] \simeq \mathbb{C}^2$. In the space $V \otimes V$ with the basis $e_{km} := e_k \otimes e_m$ ordered as follows $e_{00},\ e_{01},\ e_{10},\ e_{11}$, we have (see, e.g., [14, Ch. 2] for the definition of the tensor product of two operators)

$$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right).$$

The symmetric basis in the space $\mathrm{Sym}^2(V) \subset V \otimes V$ is as follows

$$e_0^s = e_{00} = e_0 \otimes e_0,\ e_1^s = e_{01} + e_{10} = e_0 \otimes e_1 + e_1 \otimes e_0,\ e_2^s = e_{11} = e_1 \otimes e_1. \tag{31}$$

The symmetric basis in the space $\mathrm{Sym}^n(V) \subset \underbrace{V \otimes ... \otimes V}$ for $n \in \mathbb{N}$ is

$$e_k^s = \frac{1}{k!(n-k)!} \sum_{\sigma \in S_{n+1}} \sigma(e_k),\ 0 \le k \le n, \quad \text{where } e_k = e_0 \otimes ... \otimes e_0 \otimes \underbrace{e_1 \otimes ... \otimes e_1}, \tag{32}$$

and $\sigma(e_{i_0} \otimes e_{i_1} \otimes ... \otimes e_{i_n}) = (e_{\sigma(i_0)} \otimes e_{\sigma(i_1)} \otimes ... \otimes e_{\sigma(i_n)})$ for $\sigma \in S_{n+1}$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) e_0 = e_0$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) e_0 = e_0 + e_1$ we have for the operator $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in the symmetric basis:

$$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) (e_0 \otimes e_1) = e_0 \otimes e_1 = e_0^s, \quad \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) (e_0 \otimes e_1 + e_1 \otimes e_0)$$

$$= e_0 \otimes (e_0 + e_1) + (e_0 + e_1) \otimes e_0 = 2(e_0 \otimes e_1) + e_0 \otimes e_0 + e_1 \otimes e_0 = 2e_0^s + e_1^s,$$

$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) (e_1 \otimes e_1) = (e_0 + e_1) \otimes (e_0 + e_1) = e_{00} + e_{01} + e_{10} + e_{11} = e_0^s + e_1^s + e_2^s$, hence the operator $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in the symmetric basis has the form

$$\left(\begin{smallmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right) = \sigma_1(1,2).$$

The proof of the relations (30) for general $n \in \mathbb{N}$ is similar, indeed we have

$$\mathrm{Sym}^n(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) e_k^s = \sum_{r=0}^{k} \left(\begin{smallmatrix} n-k \\ n-r \end{smallmatrix}\right) e_r^s.$$

This proves (30).

6 Results of I. Tuba and H. Wenzl

Consider the braid group B_3 given by the generators σ_1 and σ_2 and the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. B_3 maps surjectively onto $\mathrm{SL}(2,\mathbb{Z})$ via the map ρ given by

$$\sigma_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right). \tag{33}$$

It is easy to check that this is a homomorphism. Moreover we have

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \mapsto S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \sigma_1\sigma_2 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right), \quad \sigma_2\sigma_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right). \tag{34}$$

Example 1 [19]. Let V be a $(d+1)-$dimensional vector space with a basis labeled by $0, 1, ..., d$, and let $\lambda_0, \lambda_1, ..., \lambda_d$ be parameters satisfying $\lambda_i\lambda_{d-i} = c$ for a fixed constant c. Set $\overline{i} = d - i$. Then in [19] it is shown that a $\left[\frac{d+1}{2}\right]$ parameter family of representations of B_3 is given by the matrices

$$A = \left(\left(\begin{smallmatrix} \overline{i} \\ j \end{smallmatrix}\right) \lambda_i\right)_{ij}, \quad B = \left((-1)^{i+j} \left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right) \lambda_{\overline{i}}\right)_{ij}. \tag{35}$$

The proof consists in checking that $ABA = BAB = S$ with S being the skew-diagonal matrix defined by $s_{ij} = (\delta_{i,\overline{j}}(-1)^i\lambda_{\overline{i}})$. This in turn can be derived by the identity

$$\sum_{k=0}^{d} (-1)^{i+k} \left(\begin{smallmatrix} i \\ k \end{smallmatrix}\right) \left(\begin{smallmatrix} \overline{k} \\ \overline{j} \end{smallmatrix}\right) = (-1)^i \left(\begin{smallmatrix} d-i \\ d-j \end{smallmatrix}\right) = (-1)^i \left(\begin{smallmatrix} \overline{i} \\ \overline{j} \end{smallmatrix}\right)$$

(cf. [20, p.8 eq. (5)] see also (123) below). Another result in [19] is as follows.

Proposition 2.5 . Let V be a simple B_3 module of dimension $n = 2, 3$. Then there exist a basis for V for which σ_1 and σ_2 act as follows ($\lambda = (\lambda_k)_k$)

$$\sigma_1 \mapsto \sigma_1^\lambda := \begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{pmatrix}, \quad \sigma_2 \mapsto \sigma_2^\lambda := \begin{pmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{pmatrix} \text{ for } n = 2. \tag{36}$$

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & \lambda_1\lambda_3\lambda_2^{-1}+\lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \sigma_2 \mapsto \sigma_2^\lambda := \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\lambda_1\lambda_3\lambda_2^{-1}-\lambda_2 & \lambda_1 \end{pmatrix} \text{ for } n = 3. \tag{37}$$

Let us set $D = \sqrt{\lambda_2\lambda_3/\lambda_1\lambda_4}$. All simple modules for $n = 4$ are following:

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & (1+D^{-1}+D^{-2})\lambda_2 & (1+D^{-1}+D^{-2})\lambda_3 & \lambda_4 \\ 0 & \lambda_2 & (1+D^{-1})\lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \tag{38}$$

$$\sigma_2 \mapsto \sigma_2^\lambda = \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 & 0 & 0 \\ D\lambda_2 & -(D+1)\lambda_2 & \lambda_2 & 0 \\ -D^3\lambda_1 & (D^3+D^2+D)\lambda_1 & -(D^2+D+1)\lambda_1 & \lambda_1 \end{pmatrix}. \tag{39}$$

Let us set $\gamma = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{1/5}$. All simple modules for $n = 5$ are following:

$$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 & (1+\frac{\gamma^2}{\lambda_2\lambda_4})(\lambda_2+\frac{\gamma^3}{\lambda_3\lambda_4}) & (\frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma)(1+\frac{\lambda_1\lambda_5}{\gamma^2}) & (1+\frac{\lambda_2\lambda_4}{\gamma^2})(\lambda_3+\frac{\gamma^3}{\lambda_2\lambda_4}) & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & \lambda_2 & \frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma & \frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3+\gamma & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & 0 & \lambda_3 & \frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3 & \frac{\gamma^3}{\lambda_1\lambda_5} \\ 0 & 0 & 0 & \lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}. \tag{40}$$

The formula for σ_2^λ is not given in [19]. In section 9 we show that $\sigma_2^\lambda = C^{-1}\sigma_2^\Lambda C$ where $C = \mathrm{diag}(1, 1, 1, q^{-1}\frac{\lambda_3}{\lambda_4}, q^{-1}\frac{\lambda_3}{\lambda_5})$ and $\sigma_2^\Lambda = \Lambda^\sharp\sigma_2(q, 4)$ (see(2)–(3)).

7 Pascal's triangle

We shall rewrite the results and the proof of S.P. Humphries [11] in a slightly different form, using also the results of I. Tuba and H. Wenzl [19] in order to generalize Humphries' result to the case of a $q-$Pascal triangle. S.P. Humphries uses the representation of B_3

$$\sigma_1 \mapsto \Sigma_1, \quad \sigma_2 \mapsto \Sigma_2^{-1}$$

I. Tuba and H. Wenzl use another representation of B_3 (in the notations of S.P. Humphries)

$$\sigma_1 \mapsto \Sigma_2, \quad \sigma_2 \mapsto \Sigma_1^{-1}.$$

Obviously these two representations are isomorphic and the isomorphism is given by

$$\sigma_1 \mapsto \sigma_2^{-1} \text{ and } \sigma_2 \mapsto \sigma_1^{-1}.$$

We shall use the form of representation given by I. Tuba and H. Wenzl.

In the general case (for arbitrary $n \in \mathbb{N}$) we put, using Pascal's triangle

$$\sigma_1 \mapsto \sigma_1(1) := \sigma_1(1,n) := \Sigma_2(n), \quad \sigma_2 \mapsto \sigma_2(1) := \sigma_2(1,n) := \Sigma_1^{-1}(n), \quad (41)$$

where (see (2) and (3)) $\sigma_1(1) = (\sigma_1(1)_{km})_{0\leq k,m\leq n}$, $\sigma_2(1) = (\sigma_2(1)_{km})_{0\leq k,m\leq n}$ and

$$\sigma_1(1)_{km} = \begin{cases} C_{n-k}^{n-m}, & \text{if } 0 \leq k \leq m \leq n, \\ 0, & \text{if } 0 < m < k \leq n, \end{cases} \quad (42)$$

$$\sigma_2(1)_{km} = \begin{cases} 0, & \text{if } 0 < k < m \leq n, \\ (-1)^{k+m} C_k^m, & \text{if } 0 \leq m \leq k \leq n. \end{cases} \quad (43)$$

Theorem 8 [11,19] *For $\sigma_1(1)$ and $\sigma_2(1)$ defined by (42) and (43), $\Lambda = I$ and arbitrary $n \in \mathbb{N}$ we have*

$$\sigma_1(1)\sigma_2(1)\sigma_1(1) = \sigma_2(1)\sigma_1(1)\sigma_2(1) = S = \begin{pmatrix} 0 & ... & 0 & 0 & 1 \\ 0 & ... & 0 & -1 & 0 \\ 0 & ... & 1 & 0 & 0 \\ & ... & & & \\ (-1)^n & ... & 0 & 0 & 0 \end{pmatrix} \quad (44)$$

moreover, we have

$$\sigma_2(1) = (\sigma_1^{-1}(1))^\sharp. \quad (45)$$

PROOF. The identity (44) is equivalent with

$$\sigma_1(1)\sigma_2(1) = S\sigma_1^{-1}(1) = \sigma_2^{-1}(1)S. \quad (46)$$

We have in particular

$$\sigma_1(1)\sigma_2(1) = (\sigma^{12}_{km})_{0\le k,m\le n},\ \text{where}\ \sigma^{12}_{km} = \begin{cases} 0, & \text{if } 0\le k+m<n,\\ (-1)^{n-m}C_k^{n-m}, & \text{if } k+m\ge n, \end{cases} \tag{47}$$

and

$$\sigma_2(1)\sigma_1(1) = (\sigma^{21}_{km})_{0\le k,m\le n},\ \text{where}\ \sigma^{21}_{km} = \begin{cases} (-1)^k C^m_{n-k}, & \text{if } 0\le k+m\le n,\\ 0, & \text{if } k+m>n. \end{cases} \tag{48}$$

We have $\sigma_1(1)_{km} = C^{n-m}_{n-k}$. To prove that $\sigma_1^{-1}(1)_{km} = (-1)^{k+m}C^{n-m}_{n-k}$ we observe that

$$(\sigma_1(1)\sigma_1^{-1}(1))_{km} = \sum_{r=k}^n \sigma_1(1)_{kr}\sigma_1^{-1}(1)_{rm} = \sum_{r=k}^n C^{n-r}_{n-k}(-1)^{r+m}C^{n-m}_{n-r}$$

$$= \sum_{r=0}^n (-1)^{r+m}\binom{n-k}{n-r}\binom{n-r}{n-m} = \sum_{r=0}^n (-1)^{(n-r)+(n-m)}\binom{n-k}{n-r}\binom{n-r}{n-m} = \delta_{km},$$

(where in the latter step we have used the well-known identity (122), Section 11 below). Analogously $\sigma_2(1)_{km} = (-1)^{k+m}C^m_k$. To prove $\sigma_2^{-1}(1)_{km} = C^m_k$ we observe that

$$(\sigma_2(1)\sigma_2^{-1}(1))_{km} = \sum_{r=0}^n \sigma_2(1)_{kr}\sigma_2^{-1}(1)_{rm} = \sum_{r=0}^n (-1)^{k+r}C^r_k C^m_r =$$

$$\sum_{i=0}^n (-1)^{k+r}\binom{k}{r}\binom{r}{m} = \delta_{km},$$

(using again (122) in the last step). Further the identity (46)

$$\sigma_1(1)\sigma_2(1) = S\sigma_1^{-1}(1), \quad \sigma_1(1)\sigma_2(1) = \sigma_2^{-1}(1)S,$$

means

$$(\sigma_1(1)\sigma_2(1))_{km} = (S\sigma_1^{-1}(1))_{km} \quad \text{and} \quad (\sigma_1(1)\sigma_2(1))_{km} = (\sigma_2^{-1}(1)S)_{km}.$$

But

$$(\sigma_1(1)\sigma_2(1))_{km} = \sum_{r=k}^n \sigma_1(1)_{kr}\sigma_2(1)_{rm} = \sum_{r=k}^n C^{n-r}_{n-k}(-1)^{r+m}C^m_r.$$

Since $S = (S_{km})$, where $S_{km} = (-1)^k\delta_{k+m,n}$ (see (16)), we get

$$(S\sigma_1^{-1}(1))_{km} = S_{k,n-k}\sigma_1^{-1}(1)_{n-k,m} = (-1)^k(-1)^{n-k+m}C^{n-m}_{n-(n-k)}$$

$$= (-1)^{n+m}C^{n-m}_k,\ (\sigma_2^{-1}(1)S)_{km} = \sigma_2^{-1}(1)_{k,n-m}S_{n-m,m} = C^{n-m}_k(-1)^{n-m},$$

so

$$(S\sigma_1^{-1}(1))_{km} = (\sigma_2^{-1}(1)S)_{km}. \tag{49}$$

Finally the identity (46) is equivalent with the following

$$\sum_{r=k}^{n} C_{n-k}^{n-r}(-1)^{r+m}C_r^m = (-1)^{n-m}C_k^{n-m} \text{ or } \sum_{r=0}^{n}(-1)^{n-r}\binom{n-k}{n-r}\binom{r}{m} = \binom{k}{n-m},$$

which is easily proven (in the latter step we have used (123) below). $\square$

8 Pascal's triangle as $\exp T$

We give here some useful presentation for Pascal's (resp. $q-$Pascal's) triangle as operators of the form $\exp T$ (resp. $\exp_{(q)} T_q$) of some operators T (resp. T_q). Let us consider $\sigma_1(1,n)$ and $\sigma_1(1,n)^s$. Since by (3) we have $\sigma_1(q)_{km} = \binom{n-k}{n-m}_q = C_{n-k}^{n-m}(q)$ then by (15) we get

$$\sigma_1(1,n)_{km}^s = C_m^k(q). \tag{50}$$

In the space of infinite matrices let us consider two operators:

$$T_1 := \sum_{k\in\mathbb{Z}}(k+1)E_{kk+1},\ T_{(q)} := \sum_{k\in\mathbb{Z}}(k+1)_q E_{kk+1}, \tag{51}$$

where $(n)_q$ is defined by (10) and E_{km}, are infinite matrix with 1 at the place $k,m\in\mathbb{Z}$ and zeros elsewhere. Consider the $\exp T$ and $\exp_{(q)} T_q$ of these operators, namely

$$\exp T = \sum_{m=0}^{\infty}\frac{1}{m!}T^m,\ \exp_{(q)} T_{(q)} = \sum_{m=0}^{\infty}\frac{1}{(m)!_q}T_q^m. \tag{52}$$

Let us denote by P_n the projector from the space of all infinite matrices onto the subspace $\mathrm{Mat}(n+1,\mathbb{C}) = \{A = \sum_{0\leq k,m\leq n} a_{km}E_{km}\}$.

Lemma 9 *We have*

$$P_n \exp T_1 P_n = \sigma_1(1,n)^s, \quad P_n \exp_{(q)} T_{(q)} P_n = \sigma_1(q,n)^s. \tag{53}$$

PROOF. If we set

$$T(\nu) = \sum_{k\in\mathbb{Z}}\nu_{k+1}E_{kk+1},$$

where $\nu_k\in\mathbb{C},\ k\in\mathbb{Z}$, we then have

$$T(\nu)^m = \sum_{k\in\mathbb{Z}}\nu_{k+1}\nu_{k+2}...\nu_{k+m}E_{kk+m}.$$

Hence we get

$$\exp T_1 = \sum_{m=0}^{\infty} \frac{1}{m!} T_1^m = \sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}} \frac{(k+1)(k+2)...(k+m)}{m!} E_{kk+m}.$$

Finally $(\exp T_1)_{kk+m} = \frac{(k+1)(k+2)...(k+m)}{m!} = C_{k+m}^k = \sigma_1(1,n)_{kk+m}^s$. Similarly we have

$$\exp_{(q)} T_{(q)} = \sum_{m=0}^{\infty} \frac{1}{(m)!_q} T_{(q)}^m = \sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}} \frac{(k+1)_q(k+2)_q...(k+m)_q}{(m)!_q} E_{kk+m},$$

hence $(\exp_{(q)} T_{(q)})_{kk+m} = \frac{(k+1)_q(k+2)_q...(k+m)_q}{(m)!_q} = C_{k+m}^k(q) = \sigma_1(q,n)_{kk+m}^s$. $\square$

9 Irreducibility and equivalence of the representations

9.1 Operator irreducibility

Theorem 3 *The representation of the group B_3 defined by (22) have the following properties:*
1) for $q = 1,\ \Lambda_n = 1$, it is subspace irreducible *in arbitrary dimension $n \in \mathbb{N}$;*
2) for $q = 1,\ \Lambda_n = \mathrm{diag}(\lambda_k)_{k=0}^n \neq 1$ it is operator irreducible *if and only if for any $0 \leq r \leq \left[\frac{n}{2}\right]$ there exists $0 \leq i_0 < i_i < ... < i_r \leq n$ such that (see (24))*

$$M_{r+1r+2...n}^{i_0 i_i ... i_{n-r-1}}(F_{r,n}^s(q,\lambda)) \neq 0;$$

3) for $q \neq 1,\ \Lambda_n = 1$ it is subspace irreducible *if and only if $(n)_q \neq 0$.*
The representation has $[\frac{n+1}{2}] + 1$ free parameters.

We study the irreducibility of the representation (2) if the following cases:
1) $q = 1$ and $\Lambda = I$ (the Humphries case);
2) $q = 1$ and $\Lambda \neq I$ (the Tuba and Wenzl Example 1, Section 6);
3) $q \neq 1$ and $\Lambda = I$;
4) $q \neq 1$ and $\Lambda \neq I$.
Case 1). Let us set $T_n = \sum_{k=0}^{n-1}(n-k)E_{kk+1}$. By Lemma 9 we conclude that $\sigma_1(1,n) = \exp T_n$ and $\sigma_2(1,n) = \exp(-T_n^\sharp)$.

Remark 10 *The subalgebra $\{T_n, T_n^\sharp\}$ generated by the operators T_n and $T_n^\sharp$ coincide with the algebra* $\mathrm{Mat}(n+1,\mathbb{C})$.

Remark 11 *The algebra* $\mathrm{Mat}(n,\mathbb{C})$ *of all matrices in the space $\mathbb{C}^n$ is* **irreducible**.

We use the following two lemmas describing the commutant of the operator $S(q)\Lambda_n$ (see (16)) and the commutant of a strictly upper triangular matrix β defined as follows:

$$\beta=\sum_{k=0}^{n-1}\beta_{kk+1}E_{kk+1}=\begin{pmatrix}0&\beta_{01}&0&...&0\\0&0&\beta_{12}&0...&0\\&&&...&\\0&0&0&...&\beta_{n-1n}\\0&0&0&...&0\end{pmatrix}. \tag{54}$$

Let us fix for an operator $A=\sum_{0\leq k,m\leq n}a_{km}E_{km}$ the following decomposition

$$A=\sum_{r=-n}^{n}A_k,\quad \text{where } A_k:=\sum_{r=0}^{n-k}a_{rr+k}E_{rr+k},\quad A_{-k}:=\sum_{r=0}^{n-k}a_{r+kr}E_{r+kr},\ k\geq 0. \tag{55}$$

Lemma 12 *Let an operator $A\in \mathrm{Mat}(\mathrm{n}+1,\mathbb{C})$ commute with β defined by (54) and $\beta_{kk+1}\neq 0$ for all $0\leq k\leq n-1$. Then A is also upper triangular, moreover*

$$A=a_0I+\sum_{k=1}^{n}a_k\beta^k. \tag{56}$$

PROOF. We have

$$(\beta A)_{km}=\beta_{kk+1}a_{k+1m},\quad 0\leq k\leq n-1,\quad (\beta A)_{nm}=0,\ \ 0\leq m\leq n,$$

and

$$(A\beta)_{km}=a_{km-1}\beta_{m-1m},\quad 1\leq m\leq n,\quad (A\beta)_{k0}=0,\ \ 0\leq k\leq n.$$

Hence we have

$$\beta_{kk+1}a_{k+1m}=a_{km-1}\beta_{m-1m},\quad \text{or}\quad \beta_{k-1k}a_{km}=a_{k-1,m-1}\beta_{m-1m}, \tag{57}$$
$$\text{for}\quad 0\leq k,m-1\leq n-1,$$
$$\beta_{k-1k}a_{k0}=0,\quad 1\leq k\leq n,\quad a_{nm}\beta_{mm+1}=0,\quad 0\leq m\leq n-1. \tag{58}$$

Using (57) and (58) we conclude that $a_{km}=0$ for $0\leq m<k\leq n$. Indeed let us take $m=k$ in (57), then we get $\beta_{k-1k}a_{kk}=a_{k-1,k-1}\beta_{k-1k}$ or $a_{kk}=a_{k-1,k-1}$ hence $a_{kk}=a_{00}$ for all $0\leq k\leq n$. Finally we conclude that $A_0=a_{00}I$.

Similarly if we take $m=k+1$ in (57) we get $\beta_{k-1k}a_{kk+1}=a_{k-1k}\beta_{kk+1}$ or $\frac{a_{k-1k}}{\beta_{k-1k}}=\frac{a_{kk+1}}{\beta_{kk+1}}=:a_1$ hence $A_1=a_1\beta$. If we take $m=k+2$ in (57) we get using the relation $(\beta^2)_{kk+2}=\beta_{kk+1}\beta_{k+1k+2}$

$$\frac{a_{kk+2}}{a_{k-1k+1}}=\frac{\beta_{k+1k+2}}{\beta_{k-1k}}=\frac{(\beta^2)_{kk+2}}{(\beta^2)_{k-1k+1}},\ \text{ so }\ \frac{a_{k-1k+1}}{(\beta^2)_{k-1k+1}}=\frac{a_{kk+2}}{(\beta^2)_{kk+2}}=:a_2,$$

hence $A_2 = a_2\beta^2$. If we put $m = k + r$ we get using the relation $(\beta^r)_{kk+r} = \beta_{kk+1}...\beta_{k+r-1,k+r}$

$$\frac{a_{kk+r}}{a_{k-1,k+r-1}} = \frac{\beta_{k+r-1,k+r}}{\beta_{k-1k}} = \frac{(\beta^r)_{kk+r}}{(\beta^r)_{k-1,k+r-1}}, \text{ so } \frac{a_{k-1,k+r-1}}{(\beta^r)_{k-1,k+r-1}} = \frac{a_{kk+r}}{(\beta^r)_{kk+r}} =: a_r,$$

hence $A_r = a_r\beta^r$. This proves Lemma 12. $\square$

Lemma 13 *Let an operator* $A \in \mathrm{Mat}(\mathrm{n}+1, \mathbb{C})$ *commute with* $S(q)\Lambda$ *(see (16)), then*

$$q_{n-k}^{-1}\lambda_{n-k}a_{km} = (-1)^{k+m}q_k^{-1}a_{n-k,n-m}\lambda_m \quad \textit{where } A = (a_{km})_{0\le k,m\le n}. \tag{59}$$

PROOF. We have (see (16))

$$(S(q)\Lambda A)_{km} = S(q)_{k,n-k}\lambda_k a_{n-k,m} = (-1)^k q_k^{-1}\lambda_k a_{n-k,m}$$

and

$$(AS(q)\Lambda)_{km} = a_{k,n-m}S(q)_{n-m,m}\lambda_m = (-1)^{n-m}q_{n-m}^{-1}a_{k,n-m}\lambda_m.$$

Since $S(q)\Lambda A = AS(q)\Lambda$ we get (59). $\square$

To prove the irreducibility of the representation $\sigma_1 \mapsto \sigma_1(1,n) \quad \sigma_2 \mapsto \sigma_2(1,n)$ let us suppose that an operator A commute with $\sigma_1(1,n)$ and $\sigma_2(1,n)$. If we set $\beta := (\sigma_1(1,n) - I)_1 = \sum_{k=0}^{n-1}(n-k)E_{kk+1}$, the first term in the decomposition (55) of the operator $\sigma_1(1,n) - I$, by Lemma 9 we conclude that $\sigma_1(1,n) = \exp\beta$. Since A commutes with $\sigma_1(1,n)$ then A commutes with $\beta = \ln\sigma_1(1,n)$, where

$$\beta = \ln\sigma_1(1,n) = \sum_{r=1}^{n}\frac{(-1)^r}{r}(\sigma_1(1,n) - I)^r,$$

and since $\beta_{kk+1} = (\sigma_1(1) - I)_{kk+1} = C_{n-k}^{n-k-1} = (n-k) \neq 0$ for $0 \le k \le n-1$ we conclude by Lemma 12 that A is upper triangular, moreover

$$A = a_0 I + \sum_{k=1}^{n} a_k\beta^k = a_0 I + \sum_{0\le k<m\le n} a_{km}E_{km},$$

i.e. $a_{km} = 0$ for $k > m$. Since A commute with $S = \sigma_1(1,n)\sigma_2(1,n)\sigma_1(1,n)$ (see (44)) by Lemma 13 we get $a_{km} = (-1)^{k+m}a_{n-m,n-k}$, so $a_{km} = 0$ for $k < m$. Finally by (56) we conclude that $A = a_0 I$. Thus the irreducibility of the representation $\sigma_1 \mapsto \sigma_1(1,n),\ \sigma_2 \mapsto \sigma_2(1,n)$ is proved in the case 1).

Remark 14 *In fact, the representation is irreducible not only in the operator sense (i.e. that only the trivial operators commute with the representations)*

but also in the usual sense (i.e. that there are no nontrivial invariant subspaces for operators of the representations). It follows from the fact that formulas

$$X \mapsto \rho_n(X) = (\sigma_1(1,n) - I)_1,\ Y \mapsto \rho_n(Y) = (\sigma_2(1,n) - I)_{-1},\ H \mapsto [\rho_n(X), \rho_n(Y)]$$

define the irreducible representation of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ (see [14, Theorem V.4.4.]). Recall [14] that $U(\mathfrak{sl}_2)$ is the associative algebra generated by three generators X, Y, H with the relations (60).

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H, \tag{60}$$

Case 2). Idea of the proof. Let A commute with $\sigma_1^\Lambda(1,n)$ and $\sigma_2^\Lambda(1,n)$ hence by Theorem 1, relation (17), A commute with $S(q)\Lambda$. By Lemma 15 (below) A is upper triangular, hence by Lemma 13 A is diagonal so $[A, \Lambda_n] = 0$, hence $[A, \sigma_1(1,n)] = [A, \sigma_2(1,n)] = 0$ and we are in the case 1), $n \in \mathbb{N}$. i.e. A is trivial.

Lemma 15 *Let an operator $A \in \mathrm{Mat}(n+1, \mathbb{C})$ commute with $\sigma_1(1,n)\Lambda_n$ where $\Lambda_n = \mathrm{diag}(\lambda_k)_{k=0}^n$ then A is also upper triangular, i.e.*

$$A = \sum_{0 \le k \le m \le n} a_{km} E_{km} \tag{61}$$

if for any $0 \le r \le \left[\frac{n}{2}\right]$ there exists $0 \le i_0 < i_i < ... < i_r \le n$ such that (24)

$$M^{i_0 i_i \ldots i_{n-r-1}}_{r+1 r+2 \ldots n}(F^s_{r,n}(1,\lambda)) \neq 0.$$

PROOF. Let $n = 1$ and $[A, \sigma_1^\Lambda] = [A, \sigma_2^\Lambda] = 0$ where

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \quad \sigma_1^\Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 & \lambda_1 \\ 0 & \lambda_1 \end{pmatrix}, \quad \sigma_2^\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ -\lambda_0 & \lambda_0 \end{pmatrix}.$$

The relation $A\sigma_1^\Lambda = \sigma_1^\Lambda A$ gives us

$$\begin{pmatrix} a_{00}\lambda_0 & a_{00}\lambda_1 + a_{01}\lambda_1 \\ a_{10}\lambda_0 & a_{10}\lambda_1 + a_{11}\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 a_{00} + \lambda_1 a_{10} & \lambda_0 a_{01} + \lambda_1 a_{11} \\ \lambda_1 a_{10} & \lambda_1 a_{11} \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 a_{10} = 0 \\ (\lambda_1 - \lambda_0) a_{10} = 0. \end{cases}$$

Since $\lambda_1 \neq 0$ hence $a_{10} = 0$.

Let $n = 2$ and $[A, \sigma_1^\Lambda] = [A, \sigma_2^\Lambda] = 0$ where

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \quad \sigma_1^\Lambda = \sigma_1(1,2)\Lambda_2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_0 & 2\lambda_1 & \lambda_2 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

The relation $\sigma_1^\Lambda A = A\sigma_1^\Lambda$ gives us

$$\begin{pmatrix} \lambda_0 a_{00} + 2\lambda_1 a_{10} + \lambda_2 a_{20} & \lambda_0 a_{01} + 2\lambda_1 a_{11} + \lambda_2 a_{21} & \lambda_0 a_{02} + 2\lambda_1 a_{12} + \lambda_2 a_{22} \\ \lambda_1 a_{10} + \lambda_2 a_{20} & \lambda_1 a_{11} + \lambda_2 a_{21} & \lambda_1 a_{12} + \lambda_2 a_{22} \\ \lambda_2 a_{20} & \lambda_2 a_{21} & \lambda_2 a_{22} \end{pmatrix} =$$

$$\begin{pmatrix} a_{00}\lambda_0 & (2a_{00}+a_{01})\lambda_1 & (a_{00}+a_{01}+a_{02})\lambda_2 \\ a_{10}\lambda_0 & (2a_{10}+a_{11})\lambda_1 & (a_{10}+a_{11}+a_{12})\lambda_2 \\ a_{20}\lambda_0 & (2a_{20}+a_{21})\lambda_1 & (a_{20}+a_{21}+a_{22})\lambda_2 \end{pmatrix}.$$

If we compare the first columns we get

$$\begin{cases} 2\lambda_1 a_{10}+\lambda_2 a_{20}=0 \\ (\lambda_1-\lambda_0)a_{10}+\lambda_2 a_{20}=0 \\ (\lambda_2-\lambda_0)a_{20}=0 \end{cases} \quad \text{or} \quad [\sigma_1(1,2)\Lambda_2-\lambda_0 I]a^{(0)}=0, \text{ where } a^{(0)}=\begin{pmatrix} 0 \\ a_{10} \\ a_{20} \end{pmatrix}.$$

Let $a^{(0)}=0$. If we compare the second columns we get

$$\begin{cases} \lambda_2 a_{21}=0 \\ (\lambda_2-\lambda_1)a_{21}=0 \end{cases} \quad \text{or} \quad [\sigma_1(1,2)\Lambda_2-\lambda_1 I]a^{(1)}=0, \text{ where } a^{(1)}=\begin{pmatrix} 0 \\ 0 \\ a_{21} \end{pmatrix}.$$

Let $n=3$ and $[A,\sigma_1^\Lambda]=[A,\sigma_2^\Lambda]=0$ where

$$A=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \sigma_1^\Lambda=\begin{pmatrix} 1&3&3&1 \\ 0&1&2&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}\begin{pmatrix} \lambda_0&0&0&0 \\ 0&\lambda_1&0&0 \\ 0&0&\lambda_2&0 \\ 0&0&0&\lambda_3 \end{pmatrix}=\begin{pmatrix} \lambda_0&3\lambda_1&3\lambda_2&\lambda_3 \\ 0&\lambda_1&2\lambda_2&\lambda_3 \\ 0&0&\lambda_2&\lambda_3 \\ 0&0&0&\lambda_3 \end{pmatrix}.$$

The relation $A\sigma_1^\Lambda=\sigma_1^\Lambda A$ gives us

$$\begin{pmatrix} a_{00}\lambda_0 & (3a_{00}+a_{01})\lambda_1 & (3a_{00}+2a_{01}+a_{02})\lambda_2 & (a_{00}+a_{01}+a_{02}+a_{03})\lambda_3 \\ a_{10}\lambda_0 & (3a_{10}+a_{11})\lambda_1 & (3a_{10}+2a_{11}+a_{12})\lambda_2 & (a_{10}+a_{11}+a_{12}+a_{13})\lambda_3 \\ a_{20}\lambda_0 & (3a_{20}+a_{21})\lambda_1 & (3a_{20}+2a_{21}+a_{22})\lambda_2 & (a_{20}+a_{21}+a_{22}+a_{23})\lambda_3 \\ a_{30}\lambda_0 & (3a_{30}+a_{31})\lambda_1 & (3a_{30}+2a_{31}+a_{32})\lambda_2 & (a_{30}+a_{31}+a_{32}+a_{33})\lambda_3 \end{pmatrix}=$$

$$\begin{pmatrix} \lambda_0 a_{00}+3\lambda_1 a_{10}+3\lambda_2 a_{20}+\lambda_3 a_{30} & \lambda_0 a_{01}+3\lambda_1 a_{11}+3\lambda_2 a_{21}+\lambda_3 a_{31} \\ \lambda_1 a_{10}+2\lambda_2 a_{20}+\lambda_3 a_{30} & \lambda_1 a_{11}+2\lambda_2 a_{21}+\lambda_3 a_{31} \\ \lambda_2 a_{20}+\lambda_3 a_{30} & \lambda_2 a_{21}+\lambda_3 a_{31} \\ \lambda_3 a_{30} & \lambda_3 a_{31} \end{pmatrix.$$

$$\begin{matrix} \lambda_0 a_{02}+3\lambda_1 a_{12}+3\lambda_2 a_{22}+\lambda_3 a_{32} & \lambda_0 a_{03}+3\lambda_1 a_{13}+3\lambda_2 a_{23}+\lambda_3 a_{33} \\ \lambda_1 a_{12}+2\lambda_2 a_{22}+\lambda_3 a_{32} & \lambda_1 a_{13}+2\lambda_2 a_{23}+\lambda_3 a_{33} \\ \lambda_2 a_{22}+\lambda_3 a_{32} & \lambda_2 a_{23}+\lambda_3 a_{33} \\ \lambda_3 a_{32} & \lambda_3 a_{32} \end{matrix}\Bigg).$$

If we compare the first columns we get

$$\begin{cases} 3\lambda_1 a_{10}+3\lambda_2 a_{20}+\lambda_3 a_{30}=0 \\ (\lambda_1-\lambda_0)a_{10}+2\lambda_2 a_{20}+\lambda_3 a_{30}=0 \\ (\lambda_2-\lambda_0)a_{20}+\lambda_3 a_{30}=0 \\ (\lambda_3-\lambda_0)a_{30}=0 \end{cases} \quad \text{or} \quad [\sigma_1(1,3)\Lambda_3-\lambda_0 I]a^{(0)}=0, \text{ where } a^{(0)}=\begin{pmatrix} 0 \\ a_{10} \\ a_{20} \\ a_{30} \end{pmatrix}.$$

Let $a^{(0)}=0$. If we compare the second columns we get the system

$$\begin{cases} 2\lambda_1 a_{21}+3\lambda_3 a_{31}=0 \\ (\lambda_2-\lambda_1)a_{21}+\lambda_3 a_{31}=0 \\ (\lambda_3-\lambda_1)a_{31}=0 \end{cases} \quad \text{or} \quad [\sigma_1(1,3)\Lambda_3-\lambda_1 I]a^{(1)}=0, \text{ where } a^{(1)}=\begin{pmatrix} 0 \\ 0 \\ a_{21} \\ a_{31} \end{pmatrix}.$$

Let $a^{(0)}=a^{(1)}=0$. If we compare the third columns we get the system

$$\begin{cases} \lambda_3 a_{32}=0 \\ (\lambda_3-\lambda_2)a_{32}=0 \end{cases} \quad \text{or} \quad [\sigma_1(1,3)\Lambda_3-\lambda_2 I]a^{(2)}=0, \text{ where } a^{(2)}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ a_{32} \end{pmatrix}.$$

For the general case $n\in\mathbb{N}$ let us consider the following equations

$$[\sigma_1(1,n)\Lambda_n-\lambda_k I]a^{(k)}, \text{ where } a^{(k)}=(0,...,a_{k+1,k},a_{k+2,k},...,a_{n,k})^t, \ 0\le k\le n-1. \tag{62}$$

To prove Lemma it is sufficient to show that all solutions of the equations (62) are trivial. We rewrite the latter equations in the following form:

$$\sigma_1^{\Lambda,k}(1,n)b^{(k)}=0,\ \text{where}\ \sigma_1^{\Lambda,k}(1,n):=[\sigma_1(1,n)-\lambda_k\Lambda_n^{-1}],\ b^{(k)}=\Lambda_n a^{(k)},\ (63)$$

$0\le k\le n-1$. If we denote

$$F_{k,n}(1,\lambda)=[\sigma_1(1,n)-\lambda_k(\Lambda_n)^{-1}]^s \qquad (64)$$

(for notation A^s see (15)) we get by Lemma 9

$$F_{k,n}(1,\lambda)=[\sigma_1(1,n)-\lambda_k(\Lambda_n)^{-1}]^s=\exp\left(\sum_{r=0}^{n-1}(r+1)E_{rr+1}\right)-\lambda_k(\Lambda_n^\sharp)^{-1}=$$

$$\begin{pmatrix} 1-\nu_0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots\\ 0 & 1-\nu_1^k & 2 & 3 & 4 & 5 & 6 & \dots\\ 0 & 0 & 1-\nu_2^k & 3 & 6 & 10 & 15 & \dots\\ 0 & 0 & 0 & 1-\nu_3^k & 4 & 10 & 20 & \dots\\ 0 & 0 & 0 & 0 & 1-\nu_4^k & 5 & 15 & \dots\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^k & 6 & \dots\\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\nu_6^k & \dots\\ & & & & \dots & & & \end{pmatrix},$$

where $\lambda_k(\Lambda_n^\sharp)^{-1}=\mathrm{diag}(\nu_r^k)_{r=0}^n$ and $\nu_r^k=\lambda_k/\lambda_{n-r}$. Let us set $(k_n):=\sigma_1(1,n)-\lambda_k(\Lambda_n)^{-1}$. **For** $n=2$ we get

$$\sigma_1^{\Lambda,k}(1,2)=\sigma_1(1,2)-\lambda_k\Lambda_2^{-1}=\left[\begin{pmatrix}1&2&1\\0&1&1\\0&0&1\end{pmatrix}-\lambda_k\begin{pmatrix}\lambda_0&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}^{-1}\right]=\begin{pmatrix}1-\nu_0^k&2&1\\0&1-\nu_1^k&1\\0&0&1-\nu_2^k\end{pmatrix}.$$

The equations (63) gives us $\sigma_1^{\Lambda,k}(1,2)b^{(k)}=0,\ k=0,1$ i.e.

$$\begin{pmatrix}0&2&1\\0&1-\nu_1^0&1\\0&0&1-\nu_2^0\end{pmatrix}\begin{pmatrix}0\\b_{10}\\b_{20}\end{pmatrix}=0,\qquad \begin{pmatrix}1-\nu_0^1&2&1\\0&0&1\\0&0&1-\nu_2^1\end{pmatrix}\begin{pmatrix}0\\0\\b_{21}\end{pmatrix}=0.$$

We see that $b^{(0)}=0$ if some of minors $M_{12}^{01}(0_2)$, $M_{12}^{02}(0_2)$, $M_{12}^{12}(0_2)$ are not 0. Since $M_2^0(0_2)=1$ we conclude that $b^{(1)}=0$. We have

$$M_{12}^{01}(0_2)=\begin{vmatrix}2&1\\1-\nu_1^0&1\end{vmatrix},\ M_{12}^{02}(0_2)=\begin{vmatrix}2&1\\0&1-\nu_2^0\end{vmatrix},\ M_{12}^{12}(0_2)=\begin{vmatrix}1-\nu_1^0&1\\0&1-\nu_2^0\end{vmatrix},\ M_2^0(1_2)=1,$$

hence

$$M_{12}^{01}(0_2)=M_{12}^{01}(F_{0,2}^s(1,\lambda)),\ M_{12}^{02}(0_2)=M_{12}^{02}(F_{0,2}^s(1,\lambda)),$$
$$M_{12}^{12}(0_2)=M_{12}^{12}(F_{0,2}^s(1,\lambda)),\ M_2^0(0_2)=M_2^0(F_{1,2}^s(1,\lambda)),$$

where $\nu^{(r)}=(\nu_k^{(r)})_{k=0}^2$, $\nu_k^{(r)}=\lambda_r/\lambda_{2-k}$ and $0\le r\le[\frac{2}{2}]=1$.

For $n=3$ we have

$$\left[\sigma_1(1,3)-\lambda_k\Lambda_3^{-1}\right]=\left[\begin{pmatrix}1&3&3&1\\0&1&2&1\\0&0&1&1\\0&0&0&1\end{pmatrix}-\lambda_k\begin{pmatrix}\lambda_0&0&0&0\\0&\lambda_1&0&0\\0&0&\lambda_2&0\\0&0&0&\lambda_3\end{pmatrix}^{-1}\right]=\begin{pmatrix}1-\nu_0^k&3&3&1\\0&1-\nu_1^k&2&1\\0&0&1-\nu_2^k&1\\0&0&0&1-\nu_3^k\end{pmatrix},$$

the equations (63) are $\sigma_1^{\Lambda,k}(1,3)b^{(k)}=0,\ k=0,1,2$ i.e.

$$\begin{pmatrix} 0 & 3 & 3 & 1\\ 0 & 1-\nu_1^0 & 2 & 1\\ 0 & 0 & 1-\nu_2^0 & 1\\ 0 & 0 & 0 & 1-\nu_3^0 \end{pmatrix}\begin{pmatrix} 0\\ b_{10}\\ b_{20}\\ b_{30}\end{pmatrix}=0,\ \begin{pmatrix} 1-\nu_0^1 & 3 & 3 & 1\\ 0 & 0 & 2 & 1\\ 0 & 0 & 1-\nu_2^1 & 1\\ 0 & 0 & 0 & 1-\nu_3^1 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ b_{21}\\ b_{31}\end{pmatrix}=0,\ \begin{pmatrix} 1-\nu_0^2 & 3 & 3 & 1\\ 0 & 1-\nu_1^2 & 2 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1-\nu_3^2 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ b_{32}\end{pmatrix}=0.$$

We see that $b^{(0)}=0$ if some of minors $M_{123}^{i_0i_1i_2}(0_3),\ 0\le i_0<i_1<i_2\le 3$ are not 0. Since $M_{23}^{01}(1_3)=\left|\begin{smallmatrix}3&1\\2&1\end{smallmatrix}\right|=1\neq 0$ (resp $M_3^0(2_3)=1\neq 0$) we conclude that $b^{(1)}=0$ (resp $b^{(2)}=0$). As before we conclude that

$$M_{123}^{i_0i_1i_2}(0_3)=M_{123}^{i_0i_1i_2}(F_{0,23}^s(1,\lambda)),\ M_{23}^{01}(1_3)=M_{23}^{01}(F_{1,3}^s(1,\lambda)),$$

where $\nu^{(r)}=(\nu_k^{(r)})_{k=0}^3,\ \nu_k^{(r)}=\lambda_r/\lambda_{3-k}$ and $0\le r\le[\frac{3}{2}]=1$.

For $n=4$ and $n=5$ we have

$$\sigma_1^\Lambda(4,\lambda_k)=\begin{pmatrix} 1-\nu_0^k & 4 & 6 & 4 & 1\\ 0 & 1-\nu_1^k & 3 & 3 & 1\\ 0 & 0 & 1-\nu_2^k & 2 & 1\\ 0 & 0 & 0 & 1-\nu_3^k & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^k \end{pmatrix},\ \sigma_1^\Lambda(5,\lambda_k)=\begin{pmatrix} 1-\nu_0^k & 5 & 10 & 10 & 5 & 1\\ 0 & 1-\nu_1^k & 4 & 6 & 4 & 1\\ 0 & 0 & 1-\nu_2^k & 3 & 3 & 1\\ 0 & 0 & 0 & 1-\nu_3^k & 2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^k & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^k \end{pmatrix},$$

the equations (63) are $\sigma_1^{\Lambda,k}(1,4)b^{(k)}=0,\ 0\le k\le 3$ i.e.

$$\begin{pmatrix} 0 & 4 & 6 & 4 & 1\\ 0 & 1-\nu_1^0 & 3 & 3 & 1\\ 0 & 0 & 1-\nu_2^0 & 2 & 1\\ 0 & 0 & 0 & 1-\nu_3^0 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^0 \end{pmatrix}\begin{pmatrix} 0\\ b_{10}\\ b_{20}\\ b_{30}\\ b_{40}\end{pmatrix}=0,\ \begin{pmatrix} 1-\nu_0^1 & 4 & 6 & 4 & 1\\ 0 & 0 & 3 & 3 & 1\\ 0 & 0 & 1-\nu_2^1 & 2 & 1\\ 0 & 0 & 0 & 1-\nu_3^1 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^1 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ b_{21}\\ b_{31}\\ b_{41}\end{pmatrix}=0,$$

$$\begin{pmatrix} 1-\nu_1^2 & 4 & 6 & 4 & 1\\ 0 & 1-\nu_1^2 & 3 & 3 & 1\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 1-\nu_3^2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^02 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ b_{32}\\ b_{42}\end{pmatrix}=0,\ \begin{pmatrix} 1-\nu_0^3 & 4 & 6 & 4 & 1\\ 0 & 1-\nu_2^3 & 3 & 3 & 1\\ 0 & 0 & 1-\nu_2^3 & 2 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^3 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ b_{43}\end{pmatrix}=0.$$

We see that $b^{(0)}=0$ if some of minors $M_{1234}^{i_0i_1i_2i_3}(0_4),\ 0\le i_0<i_1<i_2<i_3\le 4$ are not equal to 0. Similarly, we conclude that $b^{(1)}=0$ if some of minors $M_{234}^{i_0i_1i_2}(1_4),\ 0\le i_0<i_1<i_2\le 4$ are not equal to 0. Since $M_{34}^{01}(2_4)=\left|\begin{smallmatrix}4&1\\3&1\end{smallmatrix}\right|=1\neq 0$ (resp $M_4^0(3_4)=1\neq 0$) we conclude that $b^{(2)}=0$ (resp $b^{(3)}=0$) .

In general we conclude that the system of equations (63)

$$\sigma_1^{\Lambda,k}(1,n)b^{(k)}=0,\quad 0\le k\le n-1$$

has only trivial solutions $b^{(k)}=0$ if and only if for any $0\le r\le\left[\frac{n}{2}\right]$ there exists $0\le i_0<i_i<...<i_{n-r-1}\le n$ such that (see (24))

$$M_{r+1r+2...n}^{i_0i_i...i_{n-r-1}}(r_n)=M_{r+1r+2...n}^{i_0i_i...i_{n-r-1}}(F_{r,n}^s(1,\lambda))\neq 0\ \text{ where } \nu^{(r)}=(\nu_k^{(r)})_{k=0}^n,\ \nu_k^{(r)}=\frac{\lambda_r}{\lambda_{n-k}}.$$

$\square$

For $n=5$ the equations (63) are $\sigma_1^{\Lambda,k}(1,5)b^{(k)}=0,\ 0\leq k\leq 4$ i.e.

$$\begin{pmatrix} 0 & 5 & 10 & 10 & 5 & 1\\ 0 & 1-\nu_1^0 & 4 & 6 & 4 & 1\\ 0 & 0 & 1-\nu_2^0 & 3 & 3 & 1\\ 0 & 0 & 0 & 1-\nu_3^0 & 2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^0 \end{pmatrix}\begin{pmatrix} 0\\ b_{10}\\ b_{20}\\ b_{30}\\ b_{40}\\ b_{50} \end{pmatrix}=0,\quad \begin{pmatrix} 1-\nu_0^1 & 5 & 10 & 10 & 5 & 1\\ 0 & 0 & 4 & 6 & 4 & 1\\ 0 & 0 & 1-\nu_2^1 & 3 & 3 & 1\\ 0 & 0 & 0 & 1-\nu_3^1 & 2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^1 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ b_{21}\\ b_{31}\\ b_{41}\\ b_{51} \end{pmatrix}=0,$$

$$\begin{pmatrix} 1-\nu_0^2 & 5 & 10 & 10 & 5 & 1\\ 0 & 1-\nu_1^2 & 4 & 6 & 4 & 1\\ 0 & 0 & 0 & 3 & 3 & 1\\ 0 & 0 & 0 & 1-\nu_3^2 & 2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^2 & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^2 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ b_{32}\\ b_{42}\\ b_{52} \end{pmatrix}=0,\quad \begin{pmatrix} 1-\nu_0^3 & 5 & 10 & 10 & 5 & 1\\ 0 & 1-\nu_1^3 & 4 & 6 & 4 & 1\\ 0 & 0 & 1-\nu_2^3 & 3 & 3 & 1\\ 0 & 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 1-\nu_4^3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^3 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ b_{43}\\ b_{53} \end{pmatrix}=0,$$

$$\begin{pmatrix} 1-\nu_0^4 & 5 & 10 & 10 & 5 & 1\\ 0 & 1-\nu_1^4 & 4 & 6 & 4 & 1\\ 0 & 0 & 1-\nu_2^4 & 3 & 3 & 1\\ 0 & 0 & 0 & 1-\nu_3^4 & 2 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1-\nu_5^4 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ b_{54} \end{pmatrix}=0.$$

Definition 1. *We say that the values of $\Lambda_n=\mathrm{diag}(\lambda_k)_{k=0}^n$ are suspected (for reducibility) if for some $0\leq r\leq\left[\frac{n}{2}\right]$ (see (24))*

$$M_{r+1r+2\dots n}^{i_0i_i\dots i_{n-r-1}}(F_{rn}^s(1,\lambda))=0\quad \textit{for}\quad \textit{all}\ \ 0\leq i_0<i_i<\dots<i_r\leq n. \tag{65}$$

Our **aim now is to describe shortly the suspected values** of Λ_n, for example if $r=0$ we get that for all $0\leq i_0<i_i<\dots<i_{n-1}\leq n$

$$M_{12\dots n}^{i_0i_i\dots i_{n-1}}(F_{0n}^s(1,\lambda))=0\Leftrightarrow M_{12\dots n}^{01\dots n-1}(F_{0n}^s(1,\lambda))=0,\quad \text{and } M_n^n(F_{0n}^s(1,\lambda))=0. \tag{66}$$

To complete the proof of the Theorem 3 we should show that representation is operator reducible for suspected values of Λ_n .

Firstly we find the **list** of the suspected values for $q=1,\ 0\leq r\leq\left[\frac{n}{2}\right]$. **For** $n=2,\ r=0$ we see that $M_{12}^{01}(0_2),M_{12}^{02}(0_2),M_{12}^{12}(0_2)$ all are zeros if and only if $M_{12}^{01}(0_2)=0$ and $M_2^2(0_2)=0$. Since

$$D_2(\nu):=M_{12}^{01}(0_2)=1+\nu_1^0=(\lambda_0+\lambda_1)/\lambda_1,\ \text{and } M_2^2(0)=1-\nu_2^0=(\lambda_2-\lambda_0)/\lambda_2$$

we have the suspected value $\Lambda_2=\Lambda_2^{(2)}$ where

$$\Lambda_2^{(2)}=\mathrm{diag}(\lambda_0,-\lambda_0,\lambda_0)=\lambda_0\mathrm{diag}(1,-1,1),\quad \text{rep. is reducible.} \tag{67}$$

In this case we have

$$\sigma_1^\Lambda=\begin{pmatrix} 1 & -2 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1 \end{pmatrix},\quad \sigma_2^\Lambda=\begin{pmatrix} 1 & 0 & 0\\ 1 & -1 & 0\\ 1 & -2 & 1 \end{pmatrix}. \tag{68}$$

Let us denote

$$A=\begin{pmatrix} 0 & -2 & 2\\ 1 & -3 & 1\\ 2 & -2 & 0 \end{pmatrix}.$$

Remark 16 *One may verify that the operator A commute with σ_1^Λ and σ_2^Λ defined by (68). The invariant subspace $V_2=\langle e_2=(2,1,2)\rangle$ is generated by the eigenvector $e_2:=(2,1,2)$ for σ_1^Λ and σ_2^Λ i.e. $\sigma_1^\Lambda e_2=e_2$ and $\sigma_2^\Lambda e_2=e_2$. The representation is operator irreducible $\Leftrightarrow \Lambda_2\neq\lambda_0\mathrm{diag}(1,-1,1)$.*

For $n=3,\ r=0$ we see that all minors $M_{123}^{i_0i_1i_2}(0_3),\ 0\leq i_0<i_1<i_2\leq 3$ are zeros if and only if $M_{123}^{012}(0_3)=0$ and $M_3^3(0_3)=0$. By Lemma 17 we have

$$D_3^{(0)}(\nu)^*=\begin{vmatrix} 3 & 3 & 1\\ 1-\nu_1 & 2 & 1\\ 0 & 1-\nu_2 & 1\end{vmatrix}=\frac{2\lambda_0}{\lambda_1\lambda_2}(\lambda_0+\lambda_1+\lambda_2)\ \text{ and } M_3^3(0_3)=1-\nu_3^0=(\lambda_3-\lambda_0)/\lambda_3$$

hence the suspected Λ_3 is as follows $\Lambda_3^{(3)}:=\mathrm{diag}(\lambda_0,\lambda_1,\lambda_2,\lambda_0)$ with $\lambda_0+\lambda_1+\lambda_2=0$ or

$$\Lambda_3^{(3)}=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,1)\ \text{ with }\ 1+\alpha_1+\alpha_2=0.$$

For $n=4,\ r=0$ we get that all minors $M_{1234}^{i_0i_1i_2i_3}(0_4),\ 0\leq i_0<i_1<i_2<i_3\leq 4$ are zeros if and only if $M_{1234}^{0123}(0_4)=0$ and $M_4^4(0_4)=0$. By Lemma 17 we have

$$\Lambda_4^{(4)}=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_3,1)\ \text{ with }\ 1+\alpha_1+\alpha_2+\alpha_3=0.$$

For $r=1$ we get that all minors $M_{234}^{i_0i_1i_2}(1_4),\ 0\leq i_0<i_1<i_2\leq 4$ are zeros if and only if $M_{234}^{i_0i_1i_2}(1_4),\ 0\leq i_0<i_1<i_2\leq 3$ and $M_4^4(1_4)=0$.

The general rule is similar $M_{r+1r+2...n}^{i_0i_i...i_{n-r-1}}(r_n)=0,\ 0\leq i_0<i_i<...<i_{n-1}\leq n\Leftrightarrow M_{r+1r+2...n}^{i_0i_i...i_{n-r-1}}(r_n)=0,\ 0\leq i_0<i_i<...<i_{n-1}\leq n-1$ and $M_n^n(r_n)=0$. For $r=0$ and the general case $n\in\mathbb{N}$ we should calculate the following determinants:

$$D_n^{(0)}(\nu):=M_{12...n}^{01...n-1}\left[\sigma_1(1,n)-\lambda_0\Lambda_n^{-1}\right],\tag{69}$$

$$D_n^{(k)}(\nu):=M_{k+1k+2...n}^{01...k-2}\left[\sigma_1(1,n)-\lambda_k\Lambda_n^{-1}\right],\quad 0\leq k\leq[n/2].\tag{70}$$

Let us denote by (*) the conditions (see Remark 4.5) $\lambda_r\lambda_{n-r}=c,\ 1\leq r\leq n$ and by $D_n^{(k)}(\nu)^*$ the value of $D_n^{(k)}(\nu)$ under these conditions.

Lemma 17 *We have*

$$D_n^{(0)}(\nu):=M_{23...n}^{01...n-1}\left[\sigma_1(1,n)-\lambda_0\Lambda_n^{-1}\right]=1+\sum_{r=1}^{n-1}\sum_{1\leq i_1<i_2<...<i_r\leq n-1}a_{i_1i_2...i_r}\nu_{i_1}\nu_{i_2}...\nu_{i_r},$$

$$D_n^{(0)}(\nu)^*=\frac{(n-2)!\lambda_0^{n-2}}{\prod_{k=1}^{n-1}\lambda_k}\sum_{k=0}^{n-1}\lambda_k.\tag{71}$$

PROOF. For the following notion and lemma below see [1]. We define $G_m(\lambda)$ *the generalization of the characteristic polynomial* $p_C(t)=\det(tI-C)$, $t\in\mathbb{C}$ of the matrix $C\in\mathrm{Mat}(m,\mathbb{C})$:

$$G_m(\lambda)=\det C_m(\lambda),\ \lambda\in\mathbb{C}^m,\quad \text{where}\quad C_m(\lambda)=C+\sum_{k=1}^m\lambda_kE_{kk}.\tag{72}$$

We denote by

$$M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C),\ \text{(resp. } A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C)),\ 1 \le i_1 < \dots < i_r \le m,\ 1 \le j_1 < \dots < j_r \le m$$

the minors (resp. the cofactors) of the matrix C with $i_1, i_2, \dots, i_r$ rows and $j_1, j_2, \dots, j_r$ columns. By definition $A^{12\dots m}_{12\dots m}(C) = M^{\emptyset}_{\emptyset}(C) = 1$ and $M^{12\dots m}_{12\dots m}(C) = A^{\emptyset}_{\emptyset}(C) = \det C$.

Lemma 18 *For the generalized characteristic polynomial $G_m(\lambda)$ of $C \in \mathrm{Mat}(m, \mathbb{C})$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^m$ we have:* $G_m(\lambda) =$

$$\det\left(C + \sum_{k=1}^{m} \lambda_k E_{kk}\right) = \det C + \sum_{r=1}^{m} \sum_{1 \le i_1 < i_2 < \dots < i_r \le m} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r} A^{i_1 i_2 \dots i_r}_{i_1 i_2 \dots i_r}(C). \quad (73)$$

Remark 19 *If we set $\lambda_\alpha = \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r}$ where $\alpha = (i_1, i_2, \dots, i_r)$ and $A^{\alpha}_{\alpha}(C) = A^{i_1 i_2 \dots i_r}_{i_1 i_2 \dots i_r}(C),\ \lambda_\emptyset = 1,\ A^{\emptyset}_{\emptyset}(C) = \det C$ we may write (73) as follows:*

$$G_m(\lambda) = \det C_m(\lambda) = \sum_{\emptyset \subseteq \alpha \subseteq \{1,2,\dots,m\}} \lambda_\alpha A^{\alpha}_{\alpha}(C). \quad (74)$$

Denote by C_n the matrix corresponding to minor $M^{01\dots n-1}_{23\dots n}[\sigma_1(1,n)]$. Using Lemma 18 we have for $n = 2, 3, 4$ and $n = 5$ where $\nu_k = \nu_k^{(0)} = \lambda_0/\lambda_k$

$$D_2^{(0)}(\nu) = \left|\begin{smallmatrix} 2 & 1 \\ 1-\nu_1 & 1 \end{smallmatrix}\right| = \det C_2 + \nu_1 A^1_0(C_2) = 1 + \nu_1,$$

$$D_3^{(0)}(\nu) = \left|\begin{smallmatrix} 3 & 3 & 1 \\ 1-\nu_1 & 2 & 1 \\ 0 & 1-\nu_2 & 1 \end{smallmatrix}\right| = \det C_3 + \nu_1 A^1_0(C_3) + \nu_2 A^2_1(C_3) + \nu_1\nu_2 A^{12}_{01}(C_3) = 1 + 2\nu_1 + 2\nu_2 + \nu_1\nu_2,$$

$$D_4^{(0)}(\nu) = \left|\begin{smallmatrix} 4 & 6 & 4 & 1 \\ 1-\nu_1 & 3 & 3 & 1 \\ 0 & 1-\nu_2 & 2 & 1 \\ 0 & 0 & 1-\nu_3 & 1 \end{smallmatrix}\right| = \det C_4 + \nu_1 A^1_0(C_4) + \nu_2 A^2_1(C_4) + \nu_3 A^3_2(C_4) + \nu_1\nu_2 A^{12}_{01}(C_4)$$
$$+ \nu_1\nu_3 A^{13}_{02}(C_4) + \nu_2\nu_3 A^{23}_{12}(C_4) + \nu_1\nu_2\nu_3 A^{123}_{012}(C_4) =$$
$$1 + 3\nu_1 + 5\nu_2 + 3\nu_3 + 3\nu_1\nu_2 + 5\nu_1\nu_3 + 3\nu_2\nu_3 + \nu_1\nu_2\nu_3,$$

$$D_5^{(0)}(\nu) = \left|\begin{smallmatrix} 5 & 10 & 10 & 5 & 1 \\ 1-\nu_1 & 4 & 6 & 4 & 1 \\ 0 & 1-\nu_2 & 3 & 3 & 1 \\ 0 & 0 & 1-\nu_3 & 2 & 1 \\ 0 & 0 & 0 & 1-\nu_4 & 1 \end{smallmatrix}\right| = 1 + 4\nu_1 + 9\nu_2 + 9\nu_3 + 4\nu_4 +$$
$$6\nu_1\nu_2 + 16\nu_1\nu_3 + 11\nu_1\nu_4 + 11\nu_2\nu_3 + 16\nu_2\nu_4 + 6\nu_3\nu_4$$
$$+ 4\nu_2\nu_3\nu_4 + 9\nu_1\nu_3\nu_4 + 9\nu_1\nu_2\nu_4 + 4\nu_1\nu_2\nu_3 + \nu_1\nu_2\nu_3\nu_4.$$

To prove the general formulas we use Lemma 18. We have

$$D_n^{(0)}(\nu) = \det C_n + \sum_{r=1}^{n-1} \sum_{1 \le i_1 < i_2 < \dots < i_r \le m} \nu_{i_1} \nu_{i_2} \dots \nu_{i_r} A^{i_1 i_2 \dots i_r}_{i_1 - 1 i_2 - 1 \dots i_r - 1}(C_n).$$

To prove (71) we get for $n = 2, 3$

$$D_2^{(0)}(\nu) = \left|\begin{smallmatrix} 2 & 1 \\ 1-\nu_1 & 1 \end{smallmatrix}\right| = 1 + \nu_1 = 1 + \lambda_0/\lambda_1 = (\lambda_0 + \lambda_1)/\lambda_1,$$

$$D_3^{(0)}(\nu) = \begin{vmatrix} \scriptstyle 3 & \scriptstyle 3 & \scriptstyle 1 \\ \scriptstyle 1-\nu_1 & \scriptstyle 2 & \scriptstyle 1 \\ \scriptstyle 0 & \scriptstyle 1-\nu_2 & \scriptstyle 1 \end{vmatrix} = 1+2\nu_1+2\nu_2+\nu_1\nu_2 = \frac{1}{\lambda_1\lambda_2}\left(\lambda_1\lambda_2 + 2\lambda_0\lambda_2 + 2\lambda_0\lambda_1 + \lambda_0^2\right),$$

$$(*) \Rightarrow \lambda_1\lambda_2 = \lambda_0^2 \Rightarrow D_3^{(0)}(\nu)^* = \frac{2\lambda_0}{\lambda_1\lambda_2}\left(\lambda_0 + \lambda_1 + \lambda_2\right).$$

$$D_4^{(0)}(\nu) = 1 + 3\nu_1 + 5\nu_2 + 3\nu_3 + 3\nu_1\nu_2 + 5\nu_1\nu_3 + 3\nu_2\nu_3 + \nu_1\nu_2\nu_3,$$

$$D_4^{(0)}(\nu^{(0)}) = 1 + 3\frac{\lambda_0}{\lambda_1} + 5\frac{\lambda_0}{\lambda_2} + 3\frac{\lambda_0}{\lambda_3} + 3\frac{\lambda_0^2}{\lambda_1\lambda_2} + 5\frac{\lambda_0^2}{\lambda_1\lambda_3} + 3\frac{\lambda_0^2}{\lambda_2\lambda_3} + \frac{\lambda_0^3}{\lambda_1\lambda_2\lambda_3} =$$

$$\left(\lambda_1\lambda_2\lambda_3 + 3\lambda_0\lambda_2\lambda_3 + 5\lambda_0\lambda_1\lambda_3 + 3\lambda_0\lambda_1\lambda_2 + 3\lambda_0^2\lambda_3 + 5\lambda_0^2\lambda_2 + 3\lambda_0^2\lambda_1 + 3\lambda_0^3\right)/\lambda_1\lambda_2\lambda_3 =$$

(since $\lambda_0\lambda_4 = \lambda_1\lambda_3 = \lambda_2^2$ and $\lambda_0 = \lambda_4$ we get $\lambda_0^2 = \lambda_1\lambda_3 = \lambda_2^2$ so $\lambda_0 = \pm\lambda_2$. If $\lambda_0 = \lambda_2$ we get)

$$\left(\lambda_0^2(\lambda_2 \pm 3\lambda_3 + 5\lambda_0 \pm 3\lambda_1) + \lambda_0^2(3\lambda_3 + 5\lambda_2 + 3\lambda_1 + \lambda_0)\right)/\lambda_1\lambda_2\lambda_3$$

$$\frac{6\lambda_0^2}{\lambda_1\lambda_2\lambda_3}(\lambda_0 + (1\pm 1)/2\lambda_1 + \lambda_2 + (1\pm 1)/2\lambda_3).$$

$$D_4^{(0)}(\nu)^* = \frac{6\lambda_0^2}{\lambda_1\lambda_2\lambda_3}\begin{cases} (\lambda_0+\lambda_1+\lambda_2+\lambda_3) & \text{if } \lambda_0=\lambda_2 \\ (\lambda_0+\lambda_2) & \text{if } \lambda_0=-\lambda_2 \end{cases}$$

$$D_5^{(0)}(\nu) = 1 + 4\nu_1 + 9\nu_2 + 9\nu_3 + 4\nu_4 +$$
$$6\nu_1\nu_2 + 16\nu_1\nu_3 + 11\nu_1\nu_4 + 11\nu_2\nu_3 + 16\nu_2\nu_4 + 6\nu_3\nu_4$$
$$+4\nu_2\nu_3\nu_4 + 9\nu_1\nu_3\nu_4 + 9\nu_1\nu_2\nu_4 + 4\nu_1\nu_2\nu_3 + \nu_1\nu_2\nu_3\nu_4,$$

$$D_5^{(0)}(\nu) = 1 + 4\frac{\lambda_0}{\lambda_1} + 9\frac{\lambda_0}{\lambda_2} + 9\frac{\lambda_0}{\lambda_3} + 4\frac{\lambda_0}{\lambda_4} +$$
$$6\frac{\lambda_0^2}{\lambda_1\lambda_2} + 16\frac{\lambda_0^2}{\lambda_1\lambda_3} + 11\frac{\lambda_0^2}{\lambda_1\lambda_4} + 11\frac{\lambda_0^2}{\lambda_2\lambda_3} + 16\frac{\lambda_0^2}{\lambda_2\lambda_4} + 6\frac{\lambda_0^2}{\lambda_3\lambda_4}$$
$$+4\frac{\lambda_0^3}{\lambda_2\lambda_3\lambda_4} + 9\frac{\lambda_0^3}{\lambda_1\lambda_3\lambda_4} + 9\frac{\lambda_0^3}{\lambda_1\lambda_2\lambda_4} + 4\frac{\lambda_0^3}{\lambda_1\lambda_2\lambda_3} + \frac{\lambda_0^4}{\lambda_1\lambda_2\lambda_3\lambda_4},$$

$$(*) \Rightarrow D_5^{(0)}(\nu)^* = \frac{24\lambda_0^3}{\lambda_1\lambda_2\lambda_3\lambda_4}\left(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\right).$$

For $n = 3$ we get (we write ν_k for ν_k^0)

$$D_3^{(0)}(\nu)^* = \frac{2\lambda_0}{\lambda_1\lambda_2}\left(\lambda_0 + \lambda_1 + \lambda_2\right) = \frac{2\lambda_0^2}{\lambda_1\lambda_2}(1 + \alpha_1 + \alpha_2) = 0$$

$$\Lambda_3 = \mathrm{diag}(1, \alpha_1, \alpha_2, 1),\ \ \alpha_1 + \alpha_2 = -1,\ \ \alpha_1\alpha_2 = 1,$$

$$\text{equation } \alpha^2 + \alpha + 1 = 0,\ \ \alpha_{1,2} = -1/2 \pm \sqrt{1/4 - 1} = (-1 \pm i\sqrt{3})/2. \quad (75)$$

$$\Lambda_3^{(3)} = \lambda_0 \mathrm{diag}(1, \alpha_1, \alpha_2, 1),\ \ \{\alpha_1, \alpha_2\} = \{\exp 2\pi i/3, \exp 4\pi i/3\}. \quad (76)$$

For $n = 4$ we get $D_4^{(0)}(\nu)^* = 6\lambda_0^2\left(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3\right)/(\lambda_1\lambda_2\lambda_3)$

Since $\Lambda_4 = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_0\lambda_4 = \lambda_1\lambda_3 = \lambda_2^2$ we have

$$\Lambda_4 = \lambda_0 \mathrm{diag}(1, \alpha_1, \alpha_2, \alpha_3, 1),\ \ \text{where } \alpha_k = \lambda_k/\lambda_0,$$

$$\alpha_1+\alpha_2+\alpha_3=-1 \text{ with } \alpha_1\alpha_3=\alpha_2^2=1.$$

Indeed we have $\alpha_2=\pm 1$. a) let $\alpha_2=1$ then we have

$$\alpha_1+\alpha_3=-2,\ \alpha_1\alpha_3=1,\ \text{equation } \alpha^2+2\alpha+1=0,\ (\alpha+1)^2=0,\ \alpha_{1,3}=-1$$

$$\text{hence } \Lambda_4^{(2)}=\lambda_0\mathrm{diag}(1,-1,1,-1,1),\quad (\sigma_1^\Lambda)^2=(\sigma_2^\Lambda)^2=1,\ \textbf{rep. is reducible}. \tag{77}$$

b) let $\alpha_2=-1$ then we have $\alpha_1+\alpha_3=0$, $\alpha_1\alpha_3=1$, equation $\alpha^2+1=0$, $\alpha_{1,3}=\pm i$,

$$\Rightarrow \Lambda_4^{(4)}=\lambda_0\mathrm{diag}(1,\pm i,-1,\mp i,1). \tag{78}$$

For $n=5$ we get $D_5^{(0)}(\nu)^*=\frac{24\lambda_0^3}{\lambda_1\lambda_2\lambda_3\lambda_4}\left(\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)$,

$$\text{since } \Lambda_5=\mathrm{diag}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) \text{ with } \lambda_0\lambda_5=\lambda_1\lambda_4=\lambda_2\lambda_3.$$

$$\Lambda_5=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_3,\alpha_4,1),\ \alpha_k=\lambda_k/\lambda_0$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=-1 \text{ with } \alpha_1\alpha_4=\alpha_2\alpha_3=1,\quad \text{reducible.}$$
$$\left\{\begin{smallmatrix}\alpha_1+\alpha_4=k,\\ \alpha_1\alpha_4=1\end{smallmatrix}\right. \left\{\begin{smallmatrix}\alpha_2+\alpha_3=-(1+k),\\ \alpha_2\alpha_3=1\end{smallmatrix}\right.$$
$$\alpha^2-k\alpha+1=0,\ \alpha^2+(1+k)\alpha+1=0,$$
$$\alpha_{1,4}=k/2\pm\sqrt{(k/2)^2-1},\ \alpha_{2,3}=-(1+k)/2\pm\sqrt{[(1+k)/2]^2-1}.$$

$$\Lambda_n=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,...,\alpha_{n-1},1),\ \sum_{k=1}^{n-1}\alpha_k=-1,\ \alpha_k\alpha_{n-k}=1,\ 1\leq k\leq n-1.$$

$$\Lambda_n=\mathrm{diag}(\lambda_0,\lambda_1,...,\lambda_{n-1},\lambda_n),\quad \lambda_0=\lambda_n,\quad \text{and}\quad \sum_{k=0}^{n-1}\lambda_k=0. \tag{79}$$

The solution in the general case are (see (67)–(78))

$$\Lambda_n=\lambda_0\mathrm{diag}(\alpha_k)_{k=0}^n,\ \alpha_k=\exp(\pm\frac{2\pi ik}{n}),\ \text{for } n=2m+1. \tag{80}$$

$$\Lambda_n=\lambda_0\mathrm{diag}(\alpha_k)_{k=0}^n,\ \alpha_k^{(0)}=\exp(\pm\frac{2\pi ik}{2m}),\ \alpha_k^{(1)}=\exp(\pm\frac{2\pi ik}{m}) \text{ for } n=2m. \tag{81}$$

For $n=2$ we have

$$\Lambda_2=\lambda_0\mathrm{diag}(1,-1,1),\ \alpha_1+\alpha_2=-1,\ \exp\left(\frac{2\pi ik}{2}\right),$$

For $n=3$ we have $\alpha_1+\alpha_2=-1$, $\alpha_1\alpha_2=1$, $\alpha^2+\alpha+1=0$

$$\Lambda_3^{(3)}=\lambda_0\mathrm{diag}(\alpha^0,\alpha,\alpha^2,\alpha^3),\ \alpha^3=1,\ \alpha\neq 1.$$

For $n=4$ we have

$$\Lambda_4=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_3,1),$$
$$\Lambda_4^{(2)}=\lambda_0\mathrm{diag}(1,-1,1,-1,1),\ \exp(\pm\frac{2\pi ik}{2}),\ 0\leq k\leq 4,$$

$$\alpha_1+\alpha_2+\alpha_3=-1,\ \alpha_1\alpha_3=\alpha_2^2=1,$$
$$a)\ \alpha_2=1,\ \alpha_1+\alpha_3=-2,\ \alpha_1\alpha_3=\alpha_2^2=1\Rightarrow\alpha^2+2\alpha+1=0$$
$$\Lambda_4^{(4)}=\lambda_0\mathrm{diag}(1,\pm i,-1,\mp i,1),\ \exp(\pm\frac{2\pi ik}{4}),\ 0\leq k\leq 4.$$
$$b)\ \alpha_2=-1,\ \alpha_1+\alpha_3=0,\ \alpha_1\alpha_3=1\Rightarrow\alpha^2+1=0.$$

For $n=5$ we have

$$\Lambda_5=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_3,\alpha_4,1),$$

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=-1,\ \text{with}\ \alpha_1\alpha_4=\alpha_2\alpha_3=1.$$
$$\Lambda_5^{(5)}(\alpha_1,\alpha_2)=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_2^{-1},\alpha_1^{-1},1),\ \alpha_1+\alpha_1^{-1}+\alpha_2+\alpha_2^{-1}=0.$$

In particular we have

$$\Lambda_5^{(5)}=\lambda_0\mathrm{diag}(\alpha^k)_{k=0}^5,\ \alpha^5=1,\ \alpha\neq 1.$$

For $n=6$ we get

$$\Lambda_6=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,1),$$

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=-1\ \text{with}\ \alpha_1\alpha_5=\alpha_2\alpha_4=\alpha_3^2=1,$$

$a)$ $\alpha_3=1,\ \Lambda_6^{(6),1}(\alpha_1,\alpha_2)=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,1,\alpha_2^{-1},\alpha_1^{-1},1),\ \alpha_1+\alpha_1^{-1}+\alpha_2+\alpha_2^{-1}=0,$

$b)$ $\alpha_3=-1,\ \Lambda_6^{(6),-1}(\alpha_1,\alpha_2)=\lambda_0\mathrm{diag}(1,\alpha_1,\alpha_2,-1,\alpha_2^{-1},\alpha_1^{-1},1),\ \alpha_1+\alpha_1^{-1}+\alpha_2+\alpha_2^{-1}=-2,$

In particular we have

$$\Lambda_6^{(6)}=\lambda_0\mathrm{diag}(\alpha^k)_{k=0}^6,\ \alpha^6=1,\ \alpha\neq 1.$$

In the **general case** we have for $n=2m+1$

$$\Lambda_{2m+1}=\lambda_0\mathrm{diag}(1,\alpha_1,...,\alpha_{2m},1),$$

$$\sum_{k=1}^{2m}\alpha_k=-1,\ \text{with}\ \alpha_k\alpha_{2m-k}=\alpha_1\alpha_{2m}=1,$$

$$\Lambda_{2m+1}^{(2m+1)}(\alpha_1,...,\alpha_m)=\lambda_0\mathrm{diag}(1,\alpha_1,...,\alpha_m,\alpha_m^{-1},...,\alpha_1^{-1},1),\ \sum_{k=1}^m(\alpha_k+\alpha_k^{-1})=0.$$

For $n=2m+2$ we have $\Lambda_{2m+2}=\lambda_0\mathrm{diag}(1,\alpha_1,...,\alpha_{2m+1},1),$

$$\sum_{k=1}^{2m+1}\alpha_k=-1,\ \text{with}\ \alpha_k\alpha_{2m+1-k}=\alpha_1\alpha_{2m+1}=1,$$

$a)$ $\Lambda_{2m+2}^{(2m+2),1}(\alpha_1,...,\alpha_2)=\lambda_0\mathrm{diag}(1,\alpha_1,...,\alpha_m,1,\alpha_m^{-1},...,\alpha_1^{-1},1),\ \sum_{k=1}^m(\alpha_k+\alpha_k^{-1})=0,$

$b)$ $\Lambda_{2m+2}^{(2m+2),-1}(\alpha_1,\alpha_2)=\lambda_0\mathrm{diag}(1,\alpha_1,...,\alpha_m,-1,\alpha_m^{-1},...,\alpha_1^{-1},1),\ \sum_{k=1}^m(\alpha_k+\alpha_k^{-1})=-2.$

In particular we have in both cases:

$$\Lambda_n^{(n)} = \lambda_0 \mathrm{diag}(\alpha^k)_{k=0}^n, \;\; \alpha^n = 1, \;\; \alpha \neq 1.$$

□

Case 3). We prove the **irreducibility** of the representation

$$\sigma_1 \mapsto \sigma_1^D = \sigma_1(q,n) D_n^\sharp(q) \quad \sigma_2 \mapsto \sigma_2^D = D_n(q)\sigma_2(q,n),$$

where (see (19)) $D_n(q) = \mathrm{diag}(q_r)_{r=0}^n$. By Lemma 20 below we show that the operator A, commuting with $\sigma_1(q,n)D_n^\sharp(q)$ is upper triangular under certain conditions. Further, by relation (17) A commute with $S(q)\Lambda$ hence by Lemma 13 A is diagonal: $A = \mathrm{diag}(a_{00}, ..., a_{nn})$. Using again the commutation $\sigma_1(q,n)\Lambda A \Lambda^{-1} = A\sigma_1(q,n)$ we get $\sigma_1(q,n)A = A\sigma_1(q,n)$, since $\Lambda A \Lambda^{-1} = A$, hence, by Lemma 9 A commute with $\beta(q) = \ln_{(q)} \sigma_1(q,n) = (\sigma_1(q,n) - I)_1$ where

$$\ln_{(q)} \sigma_1(q,n) = \sum_{r=1}^n \frac{(-1)^r}{(r)_q} (\sigma_1(q,n) - I)^r$$

and if $\beta_{kk+1}(q,n) := (\sigma_1(q,n) - I)_{kk+1} = C_{n-k}^{n-k-1}(q) \neq 0$ we conclude by Lemma 12 that A is trivial.

Definition 2. *We say that the value of q is suspected (for reducibility of the representation $\sigma^D(q,n)$) if for some $2 \leq r \leq n$ holds $(r)_q = 0$.*

Lemma 20 *Let an operator $A \in \mathrm{Mat(n+1,\mathbb{C})}$ commute with $\sigma_1(q,n)D_n^\sharp(q)$. then A is also upper triangular, i.e.*

$$A = \sum_{0 \leq k \leq m \leq n} a_{km} E_{km}. \tag{82}$$

if for any $0 \leq r \leq \left[\frac{n}{2}\right]$ there exists $0 \leq i_0 < i_i < ... < i_r \leq n$ such that

$$M^{i_0 i_i \dots i_{n-r-1}}_{r+1 r+2 \dots n}(F^s_{rn}(q,1)) \neq 0.$$

PROOF. For $n = 1$ we get $\sigma_1(q,1)D_1^\sharp(q) = \sigma_1(1,1)$ hence we are in the case 1) i.e. $q = 1$. For $n = 2$ we have (see (3) and (19))

$$\sigma_1(q,2) = \begin{pmatrix} 1 & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_2^\sharp(q) = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1(q,2)D_2^\sharp(q) = \begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} qa_{00}+(1+q)a_{10}+a_{20} & qa_{01}+(1+q)a_{11}+a_{21} & qa_{21}+(1+q)a_{12}+a_{22} \\ a_{10}+a_{20} & a_{11}+a_{21} & a_{12}+a_{22} \\ a_{20} & a_{21} & a_{22} \end{pmatrix},$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{00}q & a_{00}(1+q)+a_{01} & a_{00}+a_{01}+a_{02} \\ a_{10}q & a_{10}(1+q)+a_{11} & a_{10}+a_{11}+a_{12} \\ a_{20}q & a_{20}(1+q)+a_{21} & a_{20}+a_{21}+a_{22} \end{pmatrix}.$$

If we compare the first columns we get

$$\begin{cases} (1+q)a_{10}+a_{20}=0 \\ (1-q)a_{10}+a_{20}=0 \\ (1-q)a_{20}=0. \end{cases} \quad \text{or} \quad [\sigma_1(q,2)D_2^\sharp(q)-qI]a^{(0)}=0, \text{ where } a^{(0)}=\begin{pmatrix} 0 \\ a_{10} \\ a_{20} \end{pmatrix}.$$

Let $a^{(0)}=0$. If we compare the second columns we get

$$\begin{cases} a_{21}=0 \\ a_{21}=0 \end{cases} \quad \text{or} \quad [\sigma_1(q,2)D_2^\sharp(q)-I]a^{(1)}=0, \text{ where } a^{(1)}=\begin{pmatrix} 0 \\ 0 \\ a_{21} \end{pmatrix}.$$

By analogy for $n=3$ we have

$$[\sigma_1(q,3)D_3^\sharp(q)-q_3I]a^{(0)}=0, \quad [\sigma_1(q,3)D_3^\sharp(q)-q_2I]a^{(1)}=0,$$

$$[\sigma_1(q,3)D_3^\sharp(q)-q_1I]a^{(2)}=0,$$

where $a^{(0)}=(0,a_{10},a_{20},a_{30})^t$, $a^{(1)}=(0,0,a_{21},a_{31})^t$, $a^{(2)}=(0,0,0,a_{32})^t$. For general n we get

$$[\sigma_1(q,n)D_n^\sharp(q)-q_{n-k}I]a^{(k)}=0, \quad a^{(k)}=(0,0,...,a_{k+1,k},...,a_{nk})^t, \quad 0\le k<n.$$

To prove Lemma it is sufficient to show that all solutions of the latter equations are trivial. We rewrite the latter equations in the following forms:

$$[\sigma_1(q,n)-q_{n-k}(D_n^\sharp(q))^{-1}]b^{(k)}=0, \quad 0\le k<n, \text{ where } b^{(k)}=D_n^\sharp(q)a^{(k)}. \quad (83)$$

Set $(k_n):=\sigma_1(q,n)-q_{n-k}(D_n^\sharp(q))^{-1}$. The equations (83) **for** $n=2$ gives us

$$\sigma_1(q,2)=\begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1+q & 1 \\ 0 & 1-q & 1 \\ 0 & 0 & 1-q \end{pmatrix}\begin{pmatrix} 0 \\ b_{10} \\ b_{20} \end{pmatrix}=0, \quad \begin{pmatrix} q-1 & 1+q & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ b_{21} \end{pmatrix}=0.$$

Hence $b^{(0)}=0$ if some of minors $M_{12}^{12}(0_2)$, $M_{12}^{01}(0_2)$ or $M_{12}^{02}(0_2)$ are not zero. Further we get $b^{(1)}=0$ since $M_2^0(1_2)=1$. We have

$$M_{12}^{01}(0_2)=\begin{vmatrix} 1+q & 1 \\ 1-q & 1 \end{vmatrix}=M_{12}^{01}(F_{02}^s(q,1))=2q$$

$$M_{12}^{02}(0_2)=\begin{vmatrix} 1+q & 1 \\ 0 & 1-q \end{vmatrix}=M_{12}^{12}(F_{02}^s(q,1)), \ M_{12}^{12}(0_2)=\begin{vmatrix} 1-q & 1 \\ 0 & 1-q \end{vmatrix}=M_{12}^{12}(F_{02}^s(q,1)).$$

The **suspected case** (see Remark) is $\beta_{01}(q,2)=C_2^1(q)=1+q=0$ i.e. $q=-1$. We show later that the representation is reducible in this case. **For** $n=3$ we have $D_3^\sharp(q)=\operatorname{diag}(q^3,q,1,1)$ and

$$(k_3)=\sigma_1(q,3)-q_k(D_3^\sharp(q))^{-1}$$

$$=\begin{pmatrix} 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 1 & 1+q & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}-q_k\begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 1-q_k/q_3 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 1-q_k/q_2 & 1+q & 1 \\ 0 & 0 & 1-q_k/q_1 & 1 \\ 0 & 0 & 0 & 1-q_k/q_0 \end{pmatrix},$$

so the equations (83) for $n=3$ gives us

$$\begin{pmatrix} 0 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q^2 & 1+q & 1\\ 0 & 0 & 1-q^3 & 1\\ 0 & 0 & 0 & 1-q^3 \end{pmatrix}\begin{pmatrix} 0\\ b_{10}\\ b_{20}\\ b_{30}\end{pmatrix}=0,\quad \begin{pmatrix} 1-q^{-2} & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1+q & 1\\ 0 & 0 & 1-q & 1\\ 0 & 0 & 0 & 1-q \end{pmatrix}\begin{pmatrix} 0\\ 0\\ b_{21}\\ b_{31}\end{pmatrix}=0,$$

$$\begin{pmatrix} 1-q^{-3} & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q^{-1} & 1+q & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ b_{32}\end{pmatrix}=0.$$

We conclude that $b^{(0)}=0$ if $M_{123}^{i_0i_1i_2}(0_3)=M_{123}^{i_0i_1i_2}(F_{03}^s(q,1))\neq 0$ for some $0\leq i_0<i_1<i_2\leq 3$, further $b^{(1)}=0$ since $M_{23}^{01}(1_3)=M_{23}^{01}(F_{13}^s(q,1))=q^2\neq 0$, and $b^{(2)}=0$ since $M_3^0(2_3)=M_3^0(F_{23}^s(q,1))=1\neq 0$. We have

$$M_{123}^{123}(0_3)=\begin{vmatrix} 1-q^2 & 1+q & 1\\ 0 & 1-q^3 & 1\\ 0 & 0 & 1-q^3\end{vmatrix}=(1-q^2)(1-q^3)^2,\quad M_3^0(2)=1, \tag{84}$$

$$M_{012}^{012}(0_3)=\begin{vmatrix} 1+q+q^2 & 1+q+q^2 & 1\\ 1-q^2 & 1+q & 1\\ 0 & 1-q^3 & 1\end{vmatrix}=2q^3(1+q+q^2),\ M_{23}^{01}(1)=\begin{vmatrix} 1+q+q^2 & 1\\ 1+q & 1\end{vmatrix}=q^2. \tag{85}$$

The **suspected case** (see Remark) are $\beta_{01}(q,3)=C_3^1(q)=1+q+q^2=0$ and $\beta_{12}(q,3)=C_2^1(q)=1+q=0$ i.e. $q^3=1$ and $q^2=1,\ q\neq 1$. Finally the suspected values are $q=\alpha_1^{(3)}=\exp(2\pi i/3),\ q=\alpha_2^{(3)}=\exp(2\pi i2/3)$ and $q=\alpha_1^{(2)}=\exp(2\pi i/2)=-1$ where

$$\alpha_k^{(s)}=\exp(2\pi ik/s),\ 0\leq k\leq s,\ s=1,2,... \tag{86}$$

We show later that the representation is reducible in this case.

Since $D_4^\sharp(q)=\mathrm{diag}(q^6,q^3,q,1,1)$ and

$$(k_4)=\sigma_1(q,4)-q_k(D_4^\sharp(q))^{-1}$$

$$=\begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1 & 1+q & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}-q_k\begin{pmatrix} q^6 & 0 & 0 & 0 & 0\\ 0 & q^3 & 0 & 0 & 0\\ 0 & 0 & q & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}^{-1}=$$

$$\begin{pmatrix} 1-q_k/q_4 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 1-q_k/q_3 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1-q_k/q_2 & 1+q & 1\\ 0 & 0 & 0 & 1-q_k/q_1 & 1\\ 0 & 0 & 0 & 0 & 1-q_k/q_0\end{pmatrix},$$

the equations (83) **for** $n=4$ gives us

$$\begin{pmatrix} 0 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 1-q^3 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1-q^5 & 1+q & 1\\ 0 & 0 & 0 & 1-q^6 & 1\\ 0 & 0 & 0 & 0 & 1-q^6\end{pmatrix}\begin{pmatrix} 0\\ b_{10}\\ b_{20}\\ b_{30}\\ b_{40}\end{pmatrix}=0,$$

$$\begin{pmatrix} 1-q^{-3} & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 0 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1-q^2 & 1+q & 1\\ 0 & 0 & 0 & 1-q^3 & 1\\ 0 & 0 & 0 & 0 & 1-q^3\end{pmatrix}\begin{pmatrix} 0\\ 0\\ b_{21}\\ b_{31}\\ b_{41}\end{pmatrix}=0,$$

$$\begin{pmatrix} 1-q^{-5} & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-q^{-2} & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 0 & 1+q & 1 \\ 0 & 0 & 0 & 1-q & 1 \\ 0 & 0 & 0 & 0 & 1-q \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_{32} \\ b_{42} \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-q^{-6} & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-q^{-3} & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1-q^{-1} & 1+q & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_{43} \end{pmatrix} = 0.$$

The **suspected case** are $\beta_{01}(q,4) = C_4^1(q) = 1 + q + q^2 + q^3 = 0,\ \beta_{12}(q,4) = C_3^1(q) = 1+q+q^2 = 0$ and $\beta_{23}(q,4) = C_2^1(q) = 1+q = 0$ i.e. $q^s = 1,\ 2 \le s \le 4$ and $q \neq 1$. Finally the suspected values are $q = \alpha_k^{(s)},\ 1 \le k < s \le 4$.

For the general case $n \in \mathbb{N}$ the **suspected case** are $q^k = 1,\ 2 \le k \le n$ and $q \neq 1$ i.e. $q = \alpha_k^{(s)},\ 1 \le k < s \le n$.

□

Lemma 21 *The representation*

$$\sigma_1 \mapsto \sigma_1^D(q,n) := \sigma_1(q,n)D_n^\sharp(q) \quad \sigma_2 \mapsto \sigma_2^D(q,n) := D_n(q)\sigma_2(q,n),$$

is irreducible if and only if $(n)_q = 1 + q + ... + q^{n-1} \neq 0$.

PROOF. For $n = 2$ and $(2)_q = 1 + q = 0$ we have (see (3) and (19))

$$\sigma_1^D(q,2) = \begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \sigma_2^D(q,2) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -(1+q) & q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The vector $e_2 = (0,1,0)$ is the eigenvector for $\sigma_1^D(q,2)$ and $\sigma_2^D(q,2)$ with eigenvalue equal to 1:

$$\sigma_1^D(q,2)e_2 = e_2, \quad \sigma_2^D(q,2)e_2 = e_2.$$

hence the subspace $V_2 = \{te_2 = (0,t,0) \mid t \in \mathbb{C}\}$ is nontrivial invariant subspace for $\sigma_1^D(q,2)$ and $\sigma_2^D(q,2)$.

Let $1 + q \neq 0$. If we set $\sigma_k = \sigma_k^D(q,2),\ k = 1,2$ we get

$$\sigma_1 - I = \begin{pmatrix} q-1 & 1+q & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 - I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -(1+q) & q-1 \end{pmatrix}.$$

Since

$$(\sigma_1 - I)^2 = \begin{pmatrix} (q-1)^2 & (q-1)(1+q) & 2q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\sigma_2 - I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2q & -(1+q)(q-1) & (q-1)^2 \end{pmatrix}$$

we conclude that

$$A_1 := (\sigma_1 - I)^2 - (q-1)(\sigma_1 - I) = \left(\begin{smallmatrix} 0 & 0 & 1+q \\ 0 & 0 & 1-q \\ 0 & 0 & 0 \end{smallmatrix}\right), \tag{87}$$

$$A_2 := (\sigma_2 - I)^2 - (q-1)(\sigma_2 - I) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1-q & 0 & 0 \\ 1+q & 0 & 0 \end{smallmatrix}\right). \tag{88}$$

Further we get

$$(1+q)^{-1}A_1A_2 = \left(\begin{smallmatrix} 1+q & 0 & 0 \\ 1-q & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad (1+q)^{-1}A_2A_1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1-q \\ 0 & 0 & 1+q \end{smallmatrix}\right).$$

Finally we have 5 matrix

$$a = \left(\begin{smallmatrix} 0 & 0 & 1+q \\ 0 & 0 & 1-q \\ 0 & 0 & 0 \end{smallmatrix}\right), \, b = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1-q & 0 & 0 \\ 1+q & 0 & 0 \end{smallmatrix}\right), \, c = \left(\begin{smallmatrix} 1+q & 0 & 0 \\ 1-q & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \, d = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1-q \\ 0 & 0 & 1+q \end{smallmatrix}\right), \; qS(q) = \left(\begin{smallmatrix} 0 & 0 & q \\ 0 & -q & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right).$$

We get

$$(1+q)^{-1}(a-d) = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right), \, (1+q)^{-1}(c-b) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right),$$

hence we have

$$qS(q)(1+q)^{-1}(c-b) = \left(\begin{smallmatrix} 0 & 0 & q \\ 0 & -q & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -q & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right),$$

and

$$qS(q)(1+q)^{-1}(a-d) = \left(\begin{smallmatrix} 0 & 0 & q \\ 0 & -q & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & -q \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Since $q \neq 1$ we can obtain

$$\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) = I - \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Using again

$$\sigma_1 - I = \left(\begin{smallmatrix} q-1 & 1+q & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad \sigma_2 - I = \left(\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -(1+q) & q-1 \end{smallmatrix}\right),$$

we conclude that we can obtain the following matrices

$$\beta := \left(\begin{smallmatrix} 0 & 1+q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad -\beta^\sharp := \left(\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -(1+q) & 0 \end{smallmatrix}\right).$$

By Remark 10 two latter matrices generate $\mathrm{Mat}(3,\mathbb{C})$ if $(2)_q = 1+q \neq 0$. Using the latter and the previous matrices we can obtain

$$\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right).$$

Indeed we have

$$\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1+q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1+q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \; \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1+q & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1+q & 0 \end{smallmatrix}\right).$$

Finally we conclude that we can obtain all matrix units E_{kn}, $0 \leq k \leq 2$ so the algebra, generated by two matrices $\sigma_1^D(q,2)$ and $\sigma_2^D(q,2)$ coincides with the algebra $\mathrm{Mat}(3,\mathbb{C})$. So our representation is irreducible for $n = 2$ when $(2)_q = 1 + q \neq 0$ by the Remark 11.

Let $n = 3$ and $(3)_q = 1 + q + q^2 = 0$. Then $q = \alpha_k^{(3)} := \exp(2\pi i k/3),\ k = 1,2$ and we have

$$\sigma_1^D(q,3) = \begin{pmatrix} q^3 & q(1+q+q^2) & 1+q+q^2 & 1 \\ 0 & q & 1+q & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \alpha_3^{(k)} & 1+\alpha_3^{(k)} & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_2^D(q,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -(1+q) & q & 0 \\ -1 & (1+q+q^2) & -q(1+q+q^2) & q^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -(1+\alpha_3^{(k)}) & \alpha_3^{(k)} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, the subspace $V_3 = \{(0,t_1,t_2,0) \mid (t_1,t_2) \in \mathbb{C}^2\}$ is invariant subspace for $\sigma_1^D(q,3)$ and $\sigma_2^D(q,3)$.

Let $(3)_q = 1 + q + q^2 \neq 0$. If we set $\sigma_k = \sigma_k^D(q,3),\ k = 1,2$ we get

$$\sigma_1 - 1 = \begin{pmatrix} q^3-1 & q(1+q+q^2) & 1+q+q^2 & 1 \\ 0 & q-1 & 1+q & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 - 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & -(1+q) & q-1 & 0 \\ -1 & (1+q+q^2) & -q(1+q+q^2) & q^3-1 \end{pmatrix}.$$

To generalize expressions (87) we note the following

Remark 22 *Let P_A be the characteristic polynomial of the matrix A, in the space $\mathbb{C}^{n+1}$ with the spectra $(\lambda_k)_{k=0}^n$ i.e.*

$$P_A(x) = \prod_{k=0}^{n} (x - \lambda_k), \quad \textit{then} \quad A_1 = P_{\sigma_1}(\sigma_1)(\sigma_1 - I)^{-1}. \tag{89}$$

Indeed we have $\mathrm{Sp}\,\sigma_1 = \{q,1,1\}$, hence

$$P_{\sigma_1}(\sigma_1) = (\sigma_1 - qI)(\sigma_1 - I)(\sigma_1 - I)$$

and

$$A_1 := (\sigma_1 - I)^2 - (q-1)(\sigma_1 - I) = (\sigma_1 - qI)(\sigma_1 - I) = P_{\sigma_1}(\sigma_1)(\sigma_1 - I)^{-1}.$$

We would like to find the expression for $P_{\sigma_1}(\sigma_1)(\sigma_1 - I)^{-1}$ (when $n = 3$) in the following form (see (87))

$$P_{\sigma_1}(\sigma_1)(\sigma_1 - I)^{-1} = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & (1+q)t \\ 0 & 0 & 0 & (1-q)t \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find x and t we use the identity $(\sigma_1 - I)P_{\sigma_1}(\sigma_1)(\sigma_1 - I)^{-1} = 0$ i.e.

$$\begin{pmatrix} q^3-1 & q(1+q+q^2) & 1+q+q^2 & 1 \\ 0 & q-1 & 1+q & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & (1+q)t \\ 0 & 0 & 0 & (1-q)t \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

We have

$$0=(q^3-1)x+[q(1+q+q^2)(1+q)+(1+q+q^2)(1-q)]t=(1+q+q^2)\times$$

$$\{(q-1)x+[q(1+q)+(1-q)]t\}=(1+q+q^2)[(q-1)x+(1+q^2)t]$$

hence $x_3:=x=1+q^2,\ t_3:=t=1-q$. Before we have calculated $x_2=1+q,\ t_2=1-q$. Finally we have

$$P_{\sigma_1}(\sigma_1)(\sigma_1-I)^{-1}=\begin{pmatrix}0&0&0&1+q^2\\0&0&0&1-q^2\\0&0&0&(1-q)^2\\0&0&0&0\end{pmatrix}.$$

Let $n=4$ and $(4)_q=1+q+q^2+q^3=0$. Then $q=\alpha_k^{(4)}:=\exp(2\pi ik/4),\ k=1,2,3$ and we have

$$\begin{pmatrix}q^6&q^3(1+q)(1+q^2)&q(1+q^2)(1+q+q^2)&(1+q)(1+q^2)&1\\0&q^3&q(1+q+q^2)&1+q+q^2&1\\0&0&q&1+q&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&1\\0&q^3&q(1+q+q^2)&1+q+q^2&1\\0&0&q&1+q&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\-1&1&0&0&0\\1&-(1+q)&q&0&0\\-1&1+q+q^2&-q(1+q+q^2)&q^3&0\\1&-(1+q)(1+q^2)&q(1+q)(1+q^2)&-q^3(1+q^2)(1+q+q^2)&q^6\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\-1&1&0&0&0\\1&-(1+q)&q&0&0\\-1&1+q+q^2&-q(1+q+q^2)&q^3&0\\1&0&0&0&1\end{pmatrix}.$$

Obviously, the subspace $V_4=\{(0,t_1,t_2,t_3,0)\mid(t_1,t_2,t_3)\in\mathbb{C}^3\}$ is invariant subspace for $\sigma_1^D(q,4)$ and $\sigma_2^D(q,4)$. Let $(4)_q=1+q+q^2+q^3\neq0$. We would like to find the expression for $P_{\sigma_1}(\sigma_1)(\sigma_1-I)^{-1}$ and $n=4$ in the following form

$$P_{\sigma_1}(\sigma_1)(\sigma_1-I)^{-1}=\begin{pmatrix}0&0&0&0&x\\0&0&0&0&(1+q^2)t\\0&0&0&0&(1-q^2)t\\0&0&0&0&(1-q)^2t\\0&0&0&0&0\end{pmatrix}.$$

As before we get

$$\begin{pmatrix}q^6-1&q^3(1+q)(1+q^2)&q(1+q^2)(1+q+q^2)&(1+q)(1+q^2)&1\\0&q^3-1&q(1+q+q^2)&1+q+q^2&1\\0&0&q-1&1+q&1\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0&x\\0&0&0&0&(1+q^2)t\\0&0&0&0&(1-q^2)t\\0&0&0&0&(1-q)^2t\\0&0&0&0&0\end{pmatrix}=0,$$

hence

$$(q^6-1)x+(1+q^2)[q^3(1+q)(1+q^2)+q(1+q+q^2)(1-q^2)+(1+q)(1-q)^2]t=$$

$$(q^6-1)x+(1+q^2)[1+2q^3+q^6]t=(q^3+1)(q^3-1)x+(1+q^2)(q^3+1)^2.$$

Finally we conclude that

$$x_4:=x=(1+q^2)(1+q^3),\ t_4:=t=(1-q^3).$$

$$\begin{pmatrix}0&0&1+q\\0&0&1-q\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&0&1+q^2\\0&0&0&1-q^2\\0&0&0&(1-q)^2\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&0&0&(1+q^2)(1+q^3)\\0&0&0&0&(1+q^2)(1-q^3)\\0&0&0&0&(1-q^2)(1-q^3)\\0&0&0&0&(1-q)^2(1-q^3)\\0&0&0&0&0\end{pmatrix}.$$

For $n=3$ and $q=-1$

$$\sigma_1^D(q,3)=\begin{pmatrix} -1 & -1 & 1 & 1\\ 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},\ \sigma_2^D(q,3)=\begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 1 & 0 & -1 & 0\\ -1 & 1 & -1 & -1\end{pmatrix}.$$

We can prove as before that representation is irreducible. For general n the proof is similar. $\square$

Case 4). We prove the following lemma (see case 3)).

Lemma 23 *Let an operator $A\in \mathrm{Mat}(\mathrm{n}+1,\mathbb{C})$ commute with $\sigma_1(q)D_n^\sharp(q)\Lambda_n$ where $\Lambda_n=\mathrm{diag}(\lambda_0,\lambda_1,...,\lambda_n)$ with $\lambda_r\lambda_{n-r}=c,\ 0\le r\le n$ then A is also upper triangular, i.e.*

$$A=\sum_{0\le k\le m\le n} a_{km}E_{km}. \tag{90}$$

if for any $0\le r\le \left[\frac{n}{2}\right]$ there exists $0\le i_0<i_i<...<i_r\le n$ such that

$$M^{i_0 i_i\dots i_{n-r-1}}_{r+1 r+2\dots n}(F^s_{rn}(q,\lambda))\neq 0 \quad \textit{where } \nu^{(r)}=(\nu^{(r)}_k),\ \nu^{(r)}_k=\lambda_r/\lambda_{n-k}.$$

PROOF. For $n=2$ we have (see (3) and (19))

$$\sigma_1(q,2)D_2^\sharp(q)\Lambda=\begin{pmatrix} q & 1+q & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \lambda_0 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix}=\begin{pmatrix} q\lambda_0 & (1+q)\lambda_1 & \lambda_2\\ 0 & \lambda_1 & \lambda_2\\ 0 & 0 & \lambda_1\end{pmatrix},$$

$$\begin{pmatrix} q\lambda_0 & (1+q)\lambda_1 & \lambda_2\\ 0 & \lambda_1 & \lambda_2\\ 0 & 0 & \lambda_1\end{pmatrix}\begin{pmatrix} a_{00} & a_{01} & a_{02}\\ a_{10} & a_{11} & a_{12}\\ a_{20} & a_{21} & a_{22}\end{pmatrix}=$$

$$\begin{pmatrix} q\lambda_0a_{00}+(1+q)\lambda_1a_{10}+\lambda_2a_{20} & q\lambda_0a_{01}+(1+q)\lambda_1a_{11}+\lambda_2a_{21} & q\lambda_0a_{21}+(1+q)\lambda_1a_{12}+\lambda_2a_{22}\\ \lambda_1a_{10}+\lambda_2a_{20} & \lambda_1a_{11}+\lambda_2a_{21} & \lambda_1a_{12}+\lambda_2a_{22}\\ \lambda_2a_{20} & \lambda_2a_{21} & \lambda_2a_{22}\end{pmatrix},$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02}\\ a_{10} & a_{11} & a_{12}\\ a_{20} & a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} q & 1+q & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \lambda_0 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix}=\begin{pmatrix} a_{00}q\lambda_0 & [a_{00}(1+q)+a_{01}]\lambda_1 & [a_{00}+a_{01}+a_{02}]\lambda_2\\ a_{10}q\lambda_0 & [a_{10}(1+q)+a_{11}]\lambda_1 & [a_{10}+a_{11}+a_{12}]\lambda_2\\ a_{20}q\lambda_0 & [a_{20}(1+q)+a_{21}]\lambda_1 & [a_{20}+a_{21}+a_{22}]\lambda_2\end{pmatrix}.$$

If we compare the first columns we get

$$\begin{cases} (1+q)\lambda_1a_{10}+\lambda_2a_{20}=0\\ (\lambda_1-q\lambda_0)a_{10}+\lambda_2a_{20}=0\\ (\lambda_2-q\lambda_0)a_{20}=0.\end{cases}\quad \text{or}\quad [\sigma_1(q,2)D_2^\sharp(q)\Lambda_2-q_2\lambda_0I]a^{(0)}=0,\ \text{where}\ a^{(0)}=\begin{pmatrix}0\\ a_{10}\\ a_{20}\end{pmatrix}.$$

Let $a^{(0)}=0$. If we compare the second columns we get

$$\begin{cases} \lambda_2a_{21}=0\\ (\lambda_2-\lambda_1)a_{21}=0\end{cases}\quad \text{or}\quad [\sigma_1(q,2)D_2^\sharp(q)\Lambda_2-q_1\lambda_1I]a^{(1)}=0,\ \text{where}\ a^{(1)}=\begin{pmatrix}0\\ 0\\ a_{21}\end{pmatrix}.$$

By analogy for $n=3$ we have

$$[\sigma_1(q,3)D_3^\sharp(q)\Lambda_3-q_3\lambda_0I]a^{(0)}=0,\quad [\sigma_1(q,3)D_3^\sharp(q)\Lambda_3-q_2\lambda_1I]a^{(0)}=0,$$

$$[\sigma_1(q,3)D_3^\sharp(q)\Lambda_3-q_1\lambda_2I]a^{(0)}=0,$$

where $a^{(0)} = (0, a_{10}, a_{20}, a_{30})^t$, $a^{(1)} = (0, 0, a_{21}, a_{31})^t$, $a^{(2)} = (0, 0, 0, a_{32})^t$.

For general n we get

$[\sigma_1(q,n)D_n^\sharp(q)\Lambda_n - q_{n-k}\lambda_k I]a^{(k)} = 0, \quad a^{(k)} = (0, 0, ..., a_{k+1,k}, ..., a_{nk})^t, \quad 0 \le k < n.$

To prove Lemma it is sufficient to show that all solutions of the latter equations are trivial.

Let us set $(k_n) := \sigma_1^{\Lambda,k}(q,n) := \sigma_1(q,n) - q_{n-k}\lambda_k(D_n^\sharp(q)\Lambda_n)^{-1}$. We rewrite the latter equations in the following forms:

$$\sigma_1^{\Lambda,k}(q,n)b^{(k)} = 0, \quad 0 \le k < n, \text{ where } b^{(k)} = D_n^\sharp(q)\Lambda_n a^{(k)}. \tag{91}$$

If we denote

$$F_{k,n}(q,\lambda) = [\sigma_1(q,n) - q_{n-k}\lambda_k(D_n^\sharp(q)\Lambda_n)^{-1}]^s \tag{92}$$

we get by Lemma 9

$$F_{k,n}(q,\lambda) = [\sigma_1(q,n) - q_{n-k}\lambda_k(D_n^\sharp(q)\Lambda_n)^{-1}]^s =$$

$$= \exp_{(q)}\left(\sum_{r=0}^{n-1}(r+1)_q E_{rr+1}\right) - q_{n-k}\lambda_k(D_n(q)\Lambda_n^\sharp)^{-1}$$

The equations (83) for $n = 2$ gives us (we set $\nu_m^k = \lambda_k/\lambda_m$) $(0_2)b^{(0)} = 0$ and $(1_2)b^{(1)} = 0$ or

$$\begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1+q & 1 \\ 0 & 1-q\nu_1^0 & 1 \\ 0 & 0 & 1-q\nu_2^0 \end{pmatrix}\begin{pmatrix} 0 \\ b_{10} \\ b_{20} \end{pmatrix} = 0, \quad \begin{pmatrix} 1-q\nu_0^1 & 1+q & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1-\nu_2^1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ b_{21} \end{pmatrix} = 0.$$

Hence $b^{(0)} = 0$ if $M_{12}^{i_0 i_1}(0_2) \ne 0$ for some $0 \le i_0 < i_1 \le 2$ and $b^{(1)} = 0$ since $M_2^0(1_2) = 1$. We have

$M_{12}^{01}(0_2) = \begin{vmatrix} 1+q & 1 \\ 1-q\nu_1^0 & 1 \end{vmatrix}, \; M_{12}^{02}(0_2) = \begin{vmatrix} 1+q & 1 \\ 0 & 1-q\nu_2^0 \end{vmatrix}, \; M_{12}^{12}(0_2) = \begin{vmatrix} 1-q\nu_1^0 & 1 \\ 0 & 1-q\nu_2^0 \end{vmatrix}, \; M_2^0(1_2) = 1,$

hence

$$M_{12}^{01}(0_2) = M_{12}^{01}(F_{0,2}^s(q,\lambda)), \; M_{12}^{02}(0_2) = M_{12}^{02}(F_{0,2}^s(q,\lambda)),$$
$$M_{12}^{12}(0_2) = M_{12}^{12}(F_{0,2}^s(q,\lambda)), \; M_2^0(0_2) = M_2^0(F_{1,2}^s(q,\lambda)),$$

where $\nu^{(r)} = (\nu_k^{(r)})_{k=0}^2$, $\nu_k^{(r)} = \lambda_r/\lambda_{2-k}$ and $0 \le r \le [\frac{2}{2}] = 1$. We have

$$D_2(q,\nu) := M_{12}^{01}(0_2) = \begin{vmatrix} 1+q & 1 \\ 1-q\nu_1 & 1 \end{vmatrix} = q(1+\nu_1) = q(\lambda_1+\lambda_0)/\lambda_1.$$

$$M_2^2(0_2) = 1 - q\nu_2^0 = (\lambda_2 - q\lambda_0)\lambda_2.$$

We see that $M_{12}^{i_0 i_1}(0_2) = 0$ for all $0 \le i_0 < i_1 \le 2$ if and only if $M_{12}^{01}(0_2) = 0$ and $M_2^2(0_2)$ i.e. $\lambda_0 + \lambda_1 = 0$ and $\lambda_2 - q\lambda_0 = 0$.

Remark 24 *We note that conditions $\lambda_0+\lambda_1=0$ and $\lambda_2-q\lambda_0=0$ contradicts with conditions $\lambda_r\lambda_{n-r}=c,\ 0\leq r\leq n$ (see (22), for $n=2$ we have $\lambda_0\lambda_2=\lambda_1^2$). Indeed otherwise we have $\lambda_2=q\lambda_0$ and $\lambda_1=-\lambda_0$ hence $\lambda_1^2=\lambda_0^2$ and $\lambda_0\lambda_2=q\lambda_0^2$ so $\lambda_1^2\neq\lambda_0\lambda_2$ if $q\neq 1$.*

In the general case $n\in\mathbb{N}$ we should calculate the following determinant:

$$D_n(q,\nu):=M^{01\ldots n-1}_{23\ldots n}\left[\sigma_1(q,n)-q_n\lambda_0(D^\sharp_n(q)\Lambda_n)^{-1}\right]. \tag{93}$$

Since $D^\sharp_3(q)=\operatorname{diag}(q^3,q,1,1)$ and

$$(k_3)=\sigma_1(q,3)-q_k(D^\sharp_3(q)\Lambda)^{-1}$$

$$=\begin{pmatrix} 1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1 & 1+q & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}-q_{3-k}\lambda_k\begin{pmatrix} q^3\lambda_0 & 0 & 0 & 0\\ 0 & q\lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & \lambda_3\end{pmatrix}^{-1}=$$

$$\begin{pmatrix} 1-q_{3-k}/q_3\nu_0^k & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q_{3-k}/q_2\nu_1^k & 1+q & 1\\ 0 & 0 & 1-q_{3-k}/q_1\nu_2^k & 1\\ 0 & 0 & 0 & 1-q_{3-k}/q_0\nu_3^k\end{pmatrix},$$

the equations (83) **for** $n=3$ gives us

$$\begin{pmatrix} 0 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q^2\nu_1^0 & 1+q & 1\\ 0 & 0 & 1-q^3\nu_2^0 & 1\\ 0 & 0 & 0 & 1-q^3\nu_3^0\end{pmatrix}\begin{pmatrix}0\\ b_{10}\\ b_{20}\\ b_{30}\end{pmatrix}=0,\quad \begin{pmatrix} 1-q^{-2}\nu_0^1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1+q & 1\\ 0 & 0 & 1-q\nu_2^1 & 1\\ 0 & 0 & 0 & 1-q\nu_3^1\end{pmatrix}\begin{pmatrix}0\\ 0\\ b_{21}\\ b_{31}\end{pmatrix}=0,$$

$$\begin{pmatrix} 1-q^{-3}\nu_0^2 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q^{-1}\nu_1^2 & 1+q & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1-\nu_3^2\end{pmatrix}\begin{pmatrix}0\\ 0\\ 0\\ b_{32}\end{pmatrix}=0.$$

We have

$$M^{123}_{123}(0_3)=\begin{vmatrix} 1-q^2\nu_1^0 & 1+q & 1\\ 0 & 1-q^3\nu_2^0 & 1\\ 0 & 0 & 1-q^3\nu_3^0\end{vmatrix},\ M^{012}_{012}(0_3)=\begin{vmatrix} 1+q+q^2 & 1+q+q^2 & 1\\ 1-q^2\nu_1^0 & 1+q & 1\\ 0 & 1-q^3\nu_2^0 & 1\end{vmatrix}$$

$$M^{01}_{23}(1_3)=\begin{vmatrix} 1+q+q^2 & 1\\ 1+q & 1\end{vmatrix}=q^2,\quad M^0_3(2_3)=1,$$

hence

$$M^{i_0i_1i_2}_{123}(0_3)=M^{i_0i_1i_2}_{123}(F^s_{0,3}(q,\lambda)),\ M^{i_0i_1}_{23}(1_3)=M^{i_0i_1}_{23}(F^s_{1,3}(q,\lambda)),$$

$$M^{i_0}_3(2_3)=M^{i_0}_3(F^s_{2,3}(q,\lambda)).$$

Since $D^\sharp_4(q)=\operatorname{diag}(q^6,q^3,q,1,1)$ and

$$(k_4)=\sigma_1(q,4)-q_{4-k}\lambda_k(D^\sharp_4(q)\Lambda_4)^{-1}$$

$$=\begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1 & 1+q & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}-q_{4-k}\lambda_k\begin{pmatrix} q^6\lambda_0 & 0 & 0 & 0 & 0\\ 0 & q^3\lambda_1 & 0 & 0 & 0\\ 0 & 0 & q\lambda_2 & 0 & 0\\ 0 & 0 & 0 & \lambda_3 & 0\\ 0 & 0 & 0 & 0 & \lambda_4\end{pmatrix}^{-1}=$$

$$\begin{pmatrix} 1-\nu_0^k q_k/q_4 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-\nu_1^k q_k/q_3 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1-\nu_2^k q_k/q_2 & 1+q & 1 \\ 0 & 0 & 0 & 1-\nu_3^k q_k/q_1 & 1 \\ 0 & 0 & 0 & 0 & 1-\nu_4^k q_k/q_0 \end{pmatrix},$$

the equations (83) **for** $n=4$ gives us

$$\begin{pmatrix} 0 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-q^3\nu_1^0 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1-q^5\nu_2^0 & 1+q & 1 \\ 0 & 0 & 0 & 1-q^6\nu_3^0 & 1 \\ 0 & 0 & 0 & 0 & 1-q^6\nu_4^0 \end{pmatrix} \begin{pmatrix} 0 \\ b_{10} \\ b_{20} \\ b_{30} \\ b_{40} \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-q^{-3}\nu_0^1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 0 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1-q^2\nu_2^1 & 1+q & 1 \\ 0 & 0 & 0 & 1-q^3\nu_3^1 & 1 \\ 0 & 0 & 0 & 0 & 1-q^3\nu_4^1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b_{21} \\ b_{31} \\ b_{41} \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-q^{-5}\nu_0^2 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-q^{-2}\nu_1^2 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 0 & 1+q & 1 \\ 0 & 0 & 0 & 1-q\nu_3^2 & 1 \\ 0 & 0 & 0 & 0 & 1-q\nu_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_{32} \\ b_{42} \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-q^{-6}\nu_0^3 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1-q^{-3}\nu_1^3 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1-q^{-1}\nu_3^3 & 1+q & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1-\nu_4^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_{43} \end{pmatrix} = 0,$$

hence for $n=4$ we have

$$M_{1234}^{i_0 i_1 i_2 i_3}(0_4) = M_{1234}^{i_0 i_1 i_2 i_3}(F_{1,4}^s(q,\lambda)),\ \ M_{234}^{i_0 i_1 i_2}(1_4) = M_{234}^{i_0 i_1 i_2}(F_{1,4}^s(q,\lambda)),$$

$$M_{34}^{i_0 i_1}(2_4) = M_{23}^{i_0 i_1}(F_{2,4}^s(q,\lambda)),\ \ M_4^{i_0}(3_4) = M_4^{i_0}(F_{3,4}^s(q,\lambda)).$$

In general we conclude that the system of equations (91)

$$\sigma_1^{\Lambda,k}(q,n)b^{(k)} = 0, \quad 0 \le k \le n-1$$

has only trivial solutions $b^{(k)} = 0$ if and only if for any $0 \le r \le \left[\frac{n}{2}\right]$ there exists $0 \le i_0 < i_i < ... < i_{n-r-1} \le n$ such that (see (24))

$$M_{r+1 r+2 \ldots n}^{i_0 i_i \ldots i_{n-r-1}}(r_n) = M_{r+1 r+2 \ldots n}^{i_0 i_i \ldots i_{n-r-1}}(F_{r,n}^s(q,\lambda)) \neq 0.$$

□

Definition 3. *We say that the values of* $\Lambda_n = \mathrm{diag}(\lambda_k)_{k=0}^n$ *are suspected (for reducibility) if for some* $0 \le r \le \left[\frac{n}{2}\right]$ *(see (24))*

$$M_{r+1 r+2 \ldots n}^{i_0 i_i \ldots i_{n-r-1}}(F_{rn}^s(q,\lambda)) = 0 \ \ \textit{for all } 0 \le i_0 < i_i < ... < i_r \le n.$$

Our **aim now is to describe shortly the suspected values** of Λ_n, (see definition (65)). For example if $r=0$ we get that for all $0 \le i_0 < i_i < ... <$

$i_{n-1} \leq n$

$$M^{i_0 i_i \dots i_{n-1}}_{12\dots n}(F^s_{0n}(q,\lambda)) = 0 \Leftrightarrow M^{01\dots n-1}_{12\dots n}(F^s_{0n}(q,\lambda)) = 0, \quad \text{and } M^n_n(F^s_{0n}(q,\lambda)) = 0. \tag{94}$$

To complete the proof of the Theorem 3 we should show that representation is reducible for suspected values of Λ_n . We should calculate the determinant:

$$D_3(q,\nu) := M^{012}_{123}\left[\sigma_1(q,3) - q^3\lambda_0(D^\sharp_3(q)\Lambda_3)^{-1}\right]$$

$$= M^{012}_{123}\left[\begin{pmatrix} 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 1 & 1+q & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} - q^3\lambda_0\begin{pmatrix} q^3\lambda_0 & 0 & 0 & 0 \\ 0 & q\lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}^{-1}\right].$$

Let us denote by (*) the conditions (see (22)) $\lambda_r\lambda_{n-r} = c, \ 1 \leq r \leq n$ and by $D_n(q,\nu)^*$ the value of $D_n(q,\nu)$ under these conditions.

We have

$$D_3(q,\nu) = \begin{vmatrix} 1+q+q^2 & 1+q+q^2 & 1 \\ 1-q^2\nu_1 & 1+q & 1 \\ 0 & 1-q^3\nu_2 & 1 \end{vmatrix} = \begin{vmatrix} q+q^2(1+\nu_1) & q^2 \\ 1-q^2\nu_1 & q+q^3\nu_2 \end{vmatrix}$$

$$q^2\begin{vmatrix} 1+q(1+\nu_1) & 1 \\ 1-q^2\nu_1 & 1+q^2\nu_2 \end{vmatrix} = q^3\left[1 + (1+q)\nu_1 + q(1+q)\nu_2 + q^2\nu_1\nu_2\right].$$

$$\lambda_3 = q\lambda_0 \text{ hence } (*) \Rightarrow \ \lambda_0\lambda_3 = q\lambda_0^2 = \lambda_1\lambda_2, \ \text{ so } \nu_1\nu_2 = q^{-1},$$

$$D^*_3(q,\nu) = q^3\left[1 + q + (1+q)\nu_1 + q(1+q)\nu_2\right]$$

$$= q^3(1+q)\left[1+\nu_1+q\nu_2\right] = \frac{q^3(1+q)}{\lambda_1\lambda_2}\left[\lambda_1\lambda_2 + \lambda_0\lambda_2 + q\lambda_0\lambda_1\right]$$

$$= \frac{q^3(1+q)}{\lambda_1\lambda_2}\left[q\lambda_0^2 + \lambda_0\lambda_2 + q\lambda_0\lambda_1\right] = \frac{q^3(1+q)\lambda_0}{\lambda_1\lambda_2}\left[q\lambda_0 + q\lambda_1 + \lambda_2\right] = 0.$$

Finally we get

$$D^*_3(q,\nu) = \frac{q^4(1+q)\lambda_0^2}{\lambda_1\lambda_2}\left[1 + \alpha_1 + q^{-1}\alpha_2\right] = 0 \text{ with } \alpha_1\alpha_2 = q, \ \text{where } \alpha_k = \lambda_0/\lambda_k.$$

If we replace $\tilde\alpha_1 = \alpha_1, \ \tilde\alpha_2 = q^{-1}\alpha_2$ we get $\tilde\alpha_1 + \tilde\alpha_2 = -1, \ \tilde\alpha_1\tilde\alpha_2 = 1$. Using (75) and (76) we conclude that

$$\tilde\alpha_k = (-1 \pm i\sqrt{3})/2 = \exp(\pm\frac{2\pi i k}{3}), \ k = 1,2,$$

$$\Lambda_3 = \lambda_0 \mathrm{diag}(1, (-1 \pm i\sqrt{3})/2, q(-1 \mp i\sqrt{3})/2, q^3) =$$

$$\lambda_0 \mathrm{diag}(\alpha_0, \alpha_1, q\alpha_2, q^3\alpha_3) = \lambda_0 D_3(q)\mathrm{diag}(\alpha_k)^3_{k=0}, \quad \text{where } \alpha_k = \exp(\pm\frac{2\pi i k}{3}).$$

The latter values of the matrix Λ_3 contradicts with conditions $\lambda_r\lambda_{n-r} = c$. Indeed, $\lambda_1\lambda_2/\lambda_0^2 = \alpha_1 q\alpha_2 = q$ but $\lambda_0\lambda_3/\lambda_0^2 = \alpha_0 q^3\alpha_3 = q^3$. They coincide when $q = q^3$ or $q = \pm 1$.

For $n=4$ we have the following determinant:

$$D_4(q,\nu) := M^{0123}_{1234}\left[\sigma_1(q,4) - q^6\lambda_0(D_4^\sharp(q)\Lambda_4)^{-1}\right]$$

$$= M^{012}_{123}\left[\begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 0 & 1 & 1+q & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} - q^6\lambda_0\begin{pmatrix} q^6\lambda_0 & 0 & 0 & 0 & 0\\ 0 & q^3\lambda_1 & 0 & 0 & 0\\ 0 & 0 & q\lambda_2 & 0 & 0\\ 0 & 0 & 0 & \lambda_3 & 0\\ 0 & 0 & 0 & 0 & \lambda_4\end{pmatrix}^{-1}\right].$$

We have

$$D_4(q,\nu) = \begin{vmatrix} (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1\\ 1-q^3\nu_1 & 1+q+q^2 & 1+q+q^2 & 1\\ 0 & 1-q^5\nu_2 & 1+q & 1\\ 0 & 0 & 1-q^6\nu_3 & 1\end{vmatrix} = \begin{vmatrix} q+q^2+q^3(1+\nu_1) & q^2(1+q+q^2) & q^3\\ 1-q^3\nu_1 & q+q^2+q^5\nu_2 & q^2\\ 0 & 1-q^5\nu_2 & q+q^6\nu_3\end{vmatrix}$$

In the general case $n\in\mathbb{N}$ we should find

$$D_n(q,\nu) \text{ and } D_n(q,\nu)^*$$

and prove the reducibility.

9.2 Subspace irreducibility

Let us denote by $\sigma^\Lambda(q,n)$ the representation of B_3 defined by (22).

Problem. *To find a criteria of the subspace irreducibility for all representations $\sigma^\Lambda(q,n)$.*

We study here only some particular cases.

Theorem 4. *The representation $\sigma^\Lambda(q,n)$ is* subspace irreducible *for $n=1$ if and only if $\Lambda_1\neq\lambda_0(1,\alpha)$ where $\alpha^2-\alpha+1=0$.*

PROOF. *Reducibility* for $n=1$. The eigenvalues of $\sigma_1^\Lambda(1,1)=\begin{pmatrix}\lambda_0 & \lambda_1\\ 0 & \lambda_1\end{pmatrix}$ and $\sigma_2^\Lambda(1,1)=\begin{pmatrix}\lambda_1 & 0\\ -\lambda_0 & \lambda_0\end{pmatrix}$ are the following

$$e^{(1)}_{\lambda_0}=\begin{pmatrix}1\\0\end{pmatrix},\ e^{(1)}_{\lambda_1}=\begin{pmatrix}\lambda_1\\ \lambda_1-\lambda_0\end{pmatrix},\ e^{(2)}_{\lambda_0}=\begin{pmatrix}0\\1\end{pmatrix},\ e^{(1)}_{\lambda_1}=\begin{pmatrix}\lambda_1^{-1}-\lambda_0^{-1}\\ \lambda_1^{-1}\end{pmatrix}.$$

or in a short form

$$(e^{(1)}_{\lambda_0},e^{(1)}_{\lambda_1}) := \begin{pmatrix}1 & \lambda_1\\ 0 & \lambda_1-\lambda_0\end{pmatrix},\quad (e^{(2)}_{\lambda_0},e^{(2)}_{\lambda_1}) := \begin{pmatrix}0 & \lambda_1^{-1}-\lambda_0^{-1}\\ 1 & \lambda_1^{-1}\end{pmatrix}.$$

We see that $e^{(1)}_{\lambda_1}$ and $e^{(2)}_{\lambda_1}$ are linearly independent if and only if

$$\det{}_1(\lambda_0,\lambda_1) := \frac{1}{\lambda_0\lambda_1}\begin{vmatrix}\lambda_1 & \lambda_0-\lambda_1\\ \lambda_1-\lambda_0 & \lambda_0\end{vmatrix} = \begin{vmatrix}\lambda_1 & \lambda_1^{-1}-\lambda_0^{-1}\\ \lambda_1-\lambda_0 & \lambda_1^{-1}\end{vmatrix} = \frac{\lambda_0}{\lambda_1}+\frac{\lambda_1}{\lambda_0}-1\neq 0. \quad (95)$$

If we set $\alpha=\frac{\lambda_1}{\lambda_0}$ we get $\alpha+\alpha^{-1}-1=0$ or $\alpha^2-\alpha+1=0$ hence if

$$\Lambda_2=\lambda_0\mathrm{diag}(1,\alpha):\ \alpha^2-\alpha+1=0,\ \alpha_{1,2}=\frac{1}{2}\pm\frac{i\sqrt{3}}{2},\quad \alpha_{1,2}=\exp(\pm 2\pi i/6), \tag{96}$$

the representation is reducible.

Irreducibility. We have

$$A_1:=\sigma_1^{\Lambda}(1,1)-\lambda_0 I=\begin{pmatrix}0&\lambda_1\\0&\lambda_1-\lambda_0\end{pmatrix},\quad A_2:=\sigma_2^{\Lambda}(1,1)-\lambda_0 I=\begin{pmatrix}\lambda_1-\lambda_0&0\\-\lambda_0&0\end{pmatrix}$$

If $\lambda_1-\lambda_0=0$ we get $E_{01}=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $E_{10}=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. They generate the matrix algebra $\mathrm{Mat}(2,\mathbb{C})$ (see Remark) hence the representation is irreducible.

If $\lambda_1-\lambda_0\neq 0$ we have

$$A_1A_2=\begin{pmatrix}0&\lambda_1\\0&\lambda_1-\lambda_0\end{pmatrix}\begin{pmatrix}\lambda_1-\lambda_0&0\\-\lambda_0&0\end{pmatrix}=-\lambda_0\begin{pmatrix}\lambda_1&0\\\lambda_1-\lambda_0&0\end{pmatrix},\quad A_2=\begin{pmatrix}\lambda_1-\lambda_0&0\\-\lambda_0&0\end{pmatrix},$$

$$A_2A_1=\begin{pmatrix}\lambda_1-\lambda_0&0\\-\lambda_0&0\end{pmatrix}\begin{pmatrix}0&\lambda_1\\0&\lambda_1-\lambda_0\end{pmatrix}=\lambda_1\begin{pmatrix}0&\lambda_1-\lambda_0\\0&-\lambda_0\end{pmatrix},\quad A_1=\begin{pmatrix}0&\lambda_1\\0&\lambda_1-\lambda_0\end{pmatrix},$$

hence A_1A_2 and A_2 (resp. A_2A_1 and A_1) generate elements E_{00} and E_{10} (resp. E_{01} and E_{11}) iff $\det_1(\lambda_0,\lambda_1)\neq 0$, the representation is irreducible. $\square$

For $n=2$ *reducibility.* The eigenvalues of

$$\sigma_1^{\Lambda}(1,2)=\begin{pmatrix}\lambda_0&2\lambda_1&\lambda_2\\0&\lambda_1&\lambda_2\\0&0&\lambda_2\end{pmatrix},\ \sigma_2^{\Lambda}(1,2)=\begin{pmatrix}\lambda_2&0&0\\\lambda_1&-\lambda_1&0\\\lambda_0&-2\lambda_0&\lambda_0\end{pmatrix}$$

are as follows

$$e^{(1)}_{\lambda_0}=\begin{pmatrix}1\\0\\0\end{pmatrix},\ e^{(1)}_{\lambda_1}=\begin{pmatrix}2\lambda_1\\\lambda_1-\lambda_0\\0\end{pmatrix},\ e^{(1)}_{\lambda_2}=\begin{pmatrix}\lambda_2(\lambda_2+\lambda_1)\\\lambda_2(\lambda_2-\lambda_0)\\(\lambda_2-\lambda_1)(\lambda_2-\lambda_0)\end{pmatrix},$$

$$e^{(2)}_{\lambda_0}=\begin{pmatrix}0\\0\\1\end{pmatrix},\ e^{(2)}_{\lambda_1}=\begin{pmatrix}0\\\lambda_1^{-1}-\lambda_0^{-1}\\\lambda_1^{-1}\end{pmatrix},\ e^{(2)}_{\lambda_2}=\begin{pmatrix}(\lambda_2^{-1}-\lambda_1^{-1})(\lambda_2^{-1}-\lambda_0^{-1})\\\lambda_2^{-1}(\lambda_2^{-1}-\lambda_0^{-1})\\\lambda_2^{-1}(\lambda_2^{-1}+\lambda_1^{-1})\end{pmatrix}$$

or in the short form

$$(e^{(1)}_{\lambda_0},e^{(1)}_{\lambda_1},e^{(1)}_{\lambda_2})=\begin{pmatrix}1&2\lambda_1&\lambda_2(\lambda_2+\lambda_1)\\0&\lambda_1-\lambda_0&\lambda_2(\lambda_2-\lambda_0)\\0&0&(\lambda_2-\lambda_0)(\lambda_2-\lambda_1)\end{pmatrix},$$

$$(e^{(2)}_{\lambda_0},e^{(2)}_{\lambda_1},e^{(2)}_{\lambda_2})=\begin{pmatrix}0&0&(\lambda_2^{-1}-\lambda_1^{-1})(\lambda_2^{-1}-\lambda_0^{-1})\\0&\lambda_1^{-1}-\lambda_0^{-1}&\lambda_2^{-1}(\lambda_2^{-1}-\lambda_0^{-1})\\1&\lambda_1^{-1}&\lambda_2^{-1}(\lambda_2^{-1}+\lambda_1^{-1})\end{pmatrix},$$

$$(e^{(2)}_{\lambda_0},e^{(2)}_{\lambda_1},e^{(2)}_{\lambda_2})_{ij}=(e^{(1)}_{\lambda_0},e^{(1)}_{\lambda_1},e^{(1)}_{\lambda_2})_{2-i,j}.$$

Let now $\lambda_2-\lambda_0=0$. To find $e^{(1)}_{\lambda_0}$ we write $(\sigma^\Lambda_1(1,2)-\lambda_0 I)a=0$ or

$$\begin{pmatrix} 0 & 2\lambda_1 & \lambda_0 \\ 0 & \lambda_1-\lambda_0 & \lambda_0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=0,\ \left\{\begin{matrix} 2\lambda_1 a_1+\lambda_0 a_2=0 \\ (\lambda_1-\lambda_0)a_1+\lambda_0 a_2=0 \end{matrix}\right.$$

We see that $(a_1,a_2)=(0,0)$ i.e. $e^{(1)}_{\lambda_0}=(1,0,0)$ iff $\lambda_1+\lambda_0\neq 0$. Indeed

$$\begin{vmatrix} 2\lambda_1 & \lambda_0 \\ \lambda_1-\lambda_0 & \lambda_0 \end{vmatrix}=\lambda_0(\lambda_1+\lambda_0)=0\Leftrightarrow\lambda_1+\lambda_0=0.$$

In this case we have

$$(e^{(1)}_{\lambda_0},e^{(1)}_{\lambda_1})=\begin{pmatrix} 1 & 2\lambda_1 \\ 0 & \lambda_1-\lambda_0 \\ 0 & 0 \end{pmatrix},\quad (e^{(2)}_{\lambda_0},e^{(2)}_{\lambda_1})=\begin{pmatrix} 0 & 0 \\ 0 & \lambda_1^{-1}-\lambda_0^{-1} \\ 1 & \lambda_1^{-1} \end{pmatrix},$$

so representation is irreducible. If $\lambda_1+\lambda_0=0$ we conclude that $(a_1,a_2)=(\lambda_0,-2\lambda_1)$ and $e^{(1),t}_{\lambda_0}=(t,\lambda_0,-2\lambda_1)$. To find $e^{(2)}_{\lambda_0}$ we write $(\sigma^\Lambda_2(1,2)-\lambda_0 I)a=0$ or

$$\begin{pmatrix} 0 & 0 & 0 \\ \lambda_1 & \lambda_1-\lambda_0 & 0 \\ \lambda_0 & -2\lambda_0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=0,\ \left\{\begin{matrix} \lambda_1 a_0+(\lambda_1-\lambda_0)a_1=0 \\ \lambda_0 a_0-2\lambda_0 a_1=0 \end{matrix}\right.$$

$e^{(2),\tau}_{\lambda_0}=(2\lambda_0,\lambda_0,\tau)$. We have

$$(e^{(1),t}_{\lambda_0},e^{(1)}_{\lambda_1})=\begin{pmatrix} t & 2\lambda_1 \\ \lambda_0 & \lambda_1-\lambda_0 \\ -2\lambda_1 & 0 \end{pmatrix},\quad (e^{(2),\tau}_{\lambda_0},e^{(2)}_{\lambda_1})=\begin{pmatrix} 2\lambda_0 & 0 \\ \lambda_0 & \lambda_1^{-1}-\lambda_0^{-1} \\ \tau & \lambda_1^{-1} \end{pmatrix}.$$

Two vectors (a,b,t) and (τ,c,d) are proportional if and only if

$$\begin{vmatrix} a & b \\ \tau & c \end{vmatrix}=0,\ \begin{vmatrix} b & t \\ c & d \end{vmatrix}=0,\ \begin{vmatrix} a & t \\ \tau & d \end{vmatrix}=0.$$

If $b\neq 0$ and $c\neq 0$ we have $\tau=\frac{ac}{b}$, $t=\frac{bd}{c}$ and hence $\begin{vmatrix} a & t \\ \tau & d \end{vmatrix}=0$. Two vectors $e^{(1),t}_{\lambda_0}=(t,\lambda_0,-2\lambda_1)$ and $e^{(2),\tau}_{\lambda_0}=(2\lambda_0,\lambda_0,\tau)$ are proportional for $t=2\lambda_0$ and $\tau=-2\lambda_1$. In general (without condition $\lambda_0\lambda_2=\lambda_1^2$) the family of two matrix $\sigma^\Lambda_1(1,2)$ and $\sigma^\Lambda_2(1,2)$ is irreducible if and only if $\lambda_2-\lambda_0=0$ and $\lambda_0+\lambda_1\neq 0$.

In our case we have $\lambda_0\lambda_2=\lambda_1^2$ hence $\lambda_0^2=\lambda_1^2$ and $\lambda_0=\pm\lambda_1$. If $\lambda_0=\lambda_1$ we get $\Lambda_2=\lambda_0\mathrm{diag}(1,1,1)$, the representation is irreducible, if $\lambda_0=-\lambda_1$ we get $\Lambda_2=\lambda_0\mathrm{diag}(1,-1,1)$, the representation is reducible.

Irreducibility. Let us denote $A_i=(\sigma^\Lambda_i(1,2)-\lambda_0 I)(\sigma^\Lambda_i(1,2)-\lambda_1 I),\ i=1,2$. We have

$$A_1=\begin{pmatrix} 0 & 0 & \lambda_2(\lambda_2+\lambda_1) \\ 0 & 0 & \lambda_2(\lambda_2-\lambda_0) \\ 0 & 0 & (\lambda_2-\lambda_1)(\lambda_2-\lambda_0) \end{pmatrix},\quad A_2=\begin{pmatrix} (\lambda_2-\lambda_1)(\lambda_2-\lambda_0) & 0 & 0 \\ -\lambda_1(\lambda_2-\lambda_0) & 0 & 0 \\ \lambda_0(\lambda_2+\lambda_1) & 0 & 0 \end{pmatrix}$$

$$A_1A_2=\lambda_2(\lambda_2+\lambda_1)\begin{pmatrix} \lambda_2(\lambda_2+\lambda_1) & 0 & 0 \\ \lambda_2(\lambda_2-\lambda_0) & 0 & 0 \\ (\lambda_2-\lambda_1)(\lambda_2-\lambda_0) & 0 & 0 \end{pmatrix},\ A_2A_1=\lambda_2(\lambda_2+\lambda_1)\begin{pmatrix} 0 & 0 & (\lambda_2-\lambda_1)(\lambda_2-\lambda_0) \\ 0 & 0 & -\lambda_1(\lambda_2-\lambda_0) \\ 0 & 0 & \lambda_0(\lambda_2+\lambda_1) \end{pmatrix}.$$

Remark 25 *For $n = 2$ the representation is subspace irreducible if and only if $\Lambda_2 = \lambda_0 \mathrm{diag}(1, 1, 1)$.*

For $n = 3$ the eigenvalues of $\sigma_1^\Lambda(1, 3)$ and $\sigma_2^\Lambda(1, 3)$ are

$$(e_{\lambda_0}^{(1)}, e_{\lambda_1}^{(1)}, e_{\lambda_2}^{(1)}, e_{\lambda_3}^{(1)}) = \begin{pmatrix} 1 & 3\lambda_1 & 3\lambda_2(\lambda_2+\lambda_1) & \lambda_3[(\lambda_3+\lambda_2)(\lambda_3+\lambda_1)+\lambda_3(\lambda_2+\lambda_1)] \\ 0 & \lambda_1-\lambda_0 & 2\lambda_2(\lambda_2-\lambda_0) & \lambda_3(\lambda_3+\lambda_2)(\lambda_3-\lambda_0) \\ 0 & 0 & (\lambda_2-\lambda_0)(\lambda_2-\lambda_1) & \lambda_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_0) \\ 0 & 0 & 0 & (\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_0) \end{pmatrix},$$

$$(e_{\lambda_0}^{(2)}, e_{\lambda_1}^{(2)}, e_{\lambda_2}^{(2)}, e_{\lambda_3}^{(2)})_{ij} = (e_{\lambda_0^{-1}}^{(1)}, e_{\lambda_1^{-1}}^{(1)}, e_{\lambda_2^{-1}}^{(1)}, e_{\lambda_3^{-1}}^{(1)})_{3-i,j}.$$

For $n = 4$ the eigenvalues of $\sigma_1^\Lambda(1, 4)$ and $\sigma_2^\Lambda(1, 4)$ are $(e_{\lambda_0}^{(1)}, e_{\lambda_1}^{(1)}, e_{\lambda_2}^{(1)}, e_{\lambda_2}^{(1)}) =$

$$\begin{pmatrix} 1 & 4\lambda_1 & 6\lambda_2(\lambda_2+\lambda_1) & 4\lambda_3[(\lambda_3+\lambda_2)(\lambda_3+\lambda_1)+\lambda_3(\lambda_2+\lambda_1)] & \lambda_4[(\lambda_4+\lambda_3)(\lambda_4+\lambda_2)(\lambda_4+\lambda_1)+] \\ 0 & \lambda_1-\lambda_0 & 3\lambda_2(\lambda_2-\lambda_0) & 3\lambda_3(\lambda_3+\lambda_2)(\lambda_3-\lambda_0) & \lambda_4(\lambda_4+\lambda_3)(\lambda_4+\lambda_2)(\lambda_4-\lambda_0) \\ 0 & 0 & (\lambda_2-\lambda_0)(\lambda_2-\lambda_1) & \lambda_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_0) & \lambda_4(\lambda_4+\lambda_3)(\lambda_4-\lambda_1)(\lambda_4-\lambda_0) \\ 0 & 0 & 0 & (\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_0) & \lambda_4(\lambda_4-\lambda_2)(\lambda_4-\lambda_1)(\lambda_4-\lambda_0) \\ 0 & 0 & 0 & 0 & (\lambda_4-\lambda_3)(\lambda_4-\lambda_2)(\lambda_4-\lambda_1)(\lambda_4-\lambda_0) \end{pmatrix},$$

$$(e_{\lambda_0}^{(2)}, e_{\lambda_1}^{(2)}, e_{\lambda_2}^{(2)}, e_{\lambda_3}^{(2)}, e_{\lambda_4}^{(2)})_{ij} = (e_{\lambda_0^{-1}}^{(1)}, e_{\lambda_1^{-1}}^{(1)}, e_{\lambda_2^{-1}}^{(1)}, e_{\lambda_3^{-1}}^{(1)}, e_{\lambda_4^{-1}}^{(1)})_{4-i,j}.$$

In general for different $\lambda_k,\ 0 \leq k \leq n$ we get

$$(e_{\lambda_0}^{(2)}, e_{\lambda_1}^{(2)}, ..., e_{\lambda_n}^{(2)})_{ij} = (e_{\lambda_0^{-1}}^{(1)}, e_{\lambda_1^{-1}}^{(1)}, ..., e_{\lambda_n^{-1}}^{(1)})_{n-i,j},\ \ 0 \leq i, j \leq n.$$

Remark 26 *To study the subspace irreducibility it is necessary to compare all possible subspaces generated by the eigenvalues of $\sigma_1^\Lambda(1, n)$ and by the eigenvalues of $\sigma_2^\Lambda(1, n)$.*

9.3 Subspace reducibility

Representation is irreducible in the case 1). We know only **some particular cases** in the case 2). We show that representations are reducible for suspected values of Λ_n for small n and $r = 0$ (see definition 1, p.24). Let $n = 2$ and $\Lambda_2^{(2)} = \mathrm{diag}(1, -1, 1)$. We have

$$\sigma_1^\Lambda(1, 2) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^\Lambda(1, 2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

The eigenvectors e_0 for $\sigma_1^\Lambda(1, 2)$ and f_0 for $\sigma_2^\Lambda(1, 2)$ corresponding to the eigenvalue $\lambda_0 = 1$ are the following

$$e_0^t = (t, 1, 2),\ \ f_0^\tau = (2, 1, \tau).$$

Indeed we have

$$\begin{pmatrix} 0 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \\ 2 \end{pmatrix} = 0, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ \tau \end{pmatrix} = 0.$$

We conclude that

$$\langle e_0^t = (t,1,2)\rangle = \langle f_0^\tau = (2,1,\tau)\rangle \Leftrightarrow t=\tau=2.$$

Let $n=3$. We have two suspected cases: $\Lambda_3^{(s)} = \lambda_0 \mathrm{diag}(\alpha_k^{(s)})_{k=0}^3$, $s=2,3$ where $\alpha_k^{(s)} = \exp(2\pi i k/s)$. We get for $\Lambda_2^{(2)} = \mathrm{diag}(1,-1,1-1)$

$$\sigma_1^\Lambda(1,2) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_2^\Lambda(1,2) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

The eigenvectors e_0 for $\sigma_1^\Lambda(1,3)$ and f_0 for $\sigma_2^\Lambda(1,3)$ corresponding to the eigenvalue $\lambda_0 = 1$ are the following

$$e_0^t = (t,1,1,0), \quad f_0^\tau = (0,1,1,\tau).$$

Indeed we have

$$\begin{pmatrix} 0 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} t \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ -1 & 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ \tau \end{pmatrix} = 0.$$

We conclude that

$$\langle e_0^t = (t,1,1,0)\rangle = \langle f_0^\tau = (0,1,1,\tau)\rangle \Leftrightarrow t=\tau=0.$$

Lemma 27 *In the suspected cases for $r=0$ and $n=3,4$ the representations $\sigma^\Lambda(1,n)$ is subspace irreducible.*

PROOF. For $n=3$ and $r=0$ the suspected values of Λ_3 are as follows

$$\Lambda_3 = \Lambda_3^{(2)} := \lambda_0 \mathrm{diag}(1,-1,1,-1), \text{ and } \Lambda_3 = \Lambda_3^{(3)} := \lambda_0 \mathrm{diag}(\exp 2\pi i k/3)_{k=0}^3.$$

If we set $\alpha_k = \exp(2\pi i k/3)$ and take $\lambda_0 = 1$ we get $\Lambda_3^{(3)} := \mathrm{diag}(1,\alpha_1,\alpha_2,1)$ and

$$\sigma_1^\Lambda(1,3) = \sigma_1(1,3)\Lambda_3 = \begin{pmatrix} 1 & 3\alpha_1 & 3\alpha_2 & 1 \\ 0 & \alpha_1 & 2\alpha_2 & 1 \\ 0 & 0 & \alpha_2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^\Lambda(1,3) = \Lambda_3^\sharp \sigma_2(1,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha_2 & \alpha_2 & 0 & 0 \\ \alpha_1 & -2\alpha_1 & \alpha_1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

To find the eigenvectors e_k for $\sigma_1^\Lambda(1,3)$ and f_k for $\sigma_2^\Lambda(1,3)$ we have

$$(\sigma_1^\Lambda(1,3) - I)e_0 = 0, \quad (\sigma_1^\Lambda(1,3) - \alpha_1 I)e_1 = 0, \quad (\sigma_1^\Lambda(1,3) - \alpha_2 I)e_2 = 0,$$

$$(\sigma_2^\Lambda(1,3) - I)f_0 = 0, \quad (\sigma_2^\Lambda(1,3) - \alpha_1 I)f_1 = 0, \quad (\sigma_2^\Lambda(1,3) - \alpha_2 I)f_2 = 0$$

or

$$\begin{pmatrix} 0 & 3\alpha_1 & 3\alpha_2 & 1 \\ 0 & \alpha_1 - 1 & 2\alpha_2 & 1 \\ 0 & 0 & \alpha_2 - 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0, \quad \begin{pmatrix} 1-\alpha_1 & 3\alpha_1 & 3\alpha_2 & 1 \\ 0 & 0 & 2\alpha_2 & 1 \\ 0 & 0 & \alpha_2 - \alpha_1 & 1 \\ 0 & 0 & 0 & 1-\alpha_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1-\alpha_2 & 3\alpha_1 & 3\alpha_2 & 1 \\ 0 & \alpha_1-\alpha_2 & 2\alpha_2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-\alpha_2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$

The solutions are the following

$$\begin{array}{llllll}
e_0^t=(& t, & 1+\alpha_2, & 1-\alpha_1, & (1-\alpha_1)(1-\alpha_2) &), \\
e_1=(& 3\alpha_1, & -(1-\alpha_1), & 0, & 0 &), \\
e_2=(& 3(\alpha_1-1)^{-1}, & 2\alpha_2, & \alpha_2-\alpha_1, & 0 &), \\
f_0^s=(& 3(1-\alpha_1), & 3, & 2+\alpha_1, & s &), \\
f_1=(& 0, & 0, & 1-\alpha_1, & 3 &), \\
f_2=(& 0, & 1-\alpha_1, & 2, & 1-\alpha_2 &).
\end{array}$$

or

$$\begin{array}{llllll}
e_0^t=(& t, & \exp(2\pi i10/12), & \sqrt{3}\exp(2\pi i11/12), & 3 &), \\
e_1=(& 3\exp(2\pi i4/12), & \sqrt{3}\exp(2\pi i5/12), & 0, & 0 &), \\
e_2=(& \sqrt{3}\exp(2\pi i7/12), & 2\exp(2\pi i8/12), & \sqrt{3}\exp(2\pi i9/12), & 0 &), \\
f_0^s=(& 3\sqrt{3}\exp(2\pi i11/12), & 3, & \sqrt{3}\exp(2\pi i/12), & s &), \\
f_1=(& 0, & 0, & \sqrt{3}\exp(2\pi i11/12), & 3 &), \\
f_2=(& 0, & \sqrt{3}\exp(2\pi i11/12), & 2, & \sqrt{3}\exp(2\pi i/12) &).
\end{array}$$

If we set $e(k) := \exp(2\pi ik/12)$

$$\begin{array}{llllll}
e_0^t=(& t, & e(10), & \sqrt{3}e(11), & 3 &), \\
e_1=(& 3e(4), & \sqrt{3}e(5), & 0, & 0 &), \\
e_2=(& \sqrt{3}e(7), & 2e(8), & \sqrt{3}e(9), & 0 &), \\
f_0^\tau=(& 3\sqrt{3}e(11), & 3, & \sqrt{3}e(k), & \tau &), \\
f_1=(& 0, & 0, & \sqrt{3}e(11), & 3 &), \\
f_2=(& 0, & \sqrt{3}e(11), & 2, & \sqrt{3}e(1) &),
\end{array}
\quad \text{or} \quad
\begin{array}{llllll}
e_0^t=(& t, & 1, & \sqrt{3}e(1), & 3e(2) &), \\
e_1=(& \sqrt{3}, & e(1), & 0, & 0 &), \\
e_2=(& \sqrt{3}, & 2e(1), & \sqrt{3}e(2), & 0 &), \\
f_0^\tau=(& 3, & \sqrt{3}e(1), & e(2), & \tau &), \\
f_1=(& 0, & 0, & 1, & \sqrt{3}e(1) &), \\
f_2=(& 0, & \sqrt{3}, & 2e(1), & \sqrt{3}e(2) &).
\end{array}$$

We see that there no one-dimensional invariant subspaces for $\sigma_1^\Lambda(1,3)$ and $\sigma_2^\Lambda(1,3)$. We find the two-dimensional subspace in the following form

$$V_2^t := \langle e_0^t, e_1 \rangle.$$

We can verify that $f_1 \in V_2^t$ if and only if $t = t_0 := \sqrt{3}e(-1)$. Further we get that $f_0^s \in V_2^{t_0}$ if and only if $s = s_0 = \sqrt{3}e(3)$. Finally we conclude that two dimensional subspace $V_2^{t_0}$ is invariant for $\sigma_1^\Lambda(1,3)$ and $\sigma_2^\Lambda(1,3)$ since it is generated by eigenvectors $e_0^{t_0}, e_1$ for $\sigma_1^\Lambda(1,3)$ and by eigenvectors $f_0^{s_0}, f_1$ for $\sigma_2^\Lambda(1,3)$.

Let $n = 4$. We have three suspected cases: $\Lambda_3^{(s)} = \lambda_0 \mathrm{diag}(\alpha_k^{(s)})_{k=0}^4,\ s = 2,3,4$. We get for $\Lambda_4^{(2)} = \mathrm{diag}(1,-1,1-1,1)$ (resp. for $\Lambda_4^{(3)}$ and $\Lambda_4^{(4)}$)

$$\sigma_1^\Lambda(1,3) = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & -1 & 3 & -3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^\Lambda(1,3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix},$$

$$\sigma_1^\Lambda(1,4) = \begin{pmatrix} 1 & 4\alpha_1 & 6\alpha_2 & 4 & \alpha_1 \\ 0 & \alpha_1 & 3\alpha_2 & 3 & \alpha_1 \\ 0 & 0 & \alpha_2 & 2 & \alpha_1 \\ 0 & 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 0 & \alpha_1 \end{pmatrix}, \quad \sigma_2^\Lambda(1,4) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \alpha_2 & -2\alpha_2 & \alpha_2 & 0 & 0 \\ -\alpha_1 & 3\alpha_1 & -3\alpha_1 & \alpha_1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix},$$

$$\sigma_1^\Lambda(1,4) = \begin{pmatrix} 1 & 4i & -6 & -4i & 1 \\ 0 & i & -3 & -3i & 1 \\ 0 & 0 & -1 & -2i & 1 \\ 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^\Lambda(1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -i & 3i & -3i & i & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Problem, $n = 4$**.** To find the eigenvectors $e_0^{(s),t_0}$, $e_1^{(s),t_1}$, $e_2^{(s),t_2}$ for $\sigma_1^\Lambda(1,4)$ and $f_0^{(s),\tau_0}$, $f_1^{(s),\tau_1}$, $f_2^{(s),\tau_2}$ for $\sigma_2^\Lambda(1,4)$ in three different cases $\Lambda_3^{(s)} = \lambda_0 \mathrm{diag}(\alpha_k^{(s)})_{k=0}^4,\ s = 2,3,4$.

Define in the case $\Lambda_3^{(2)} = \mathrm{diag}(1, -1, 1-1, 1)$

$$V_4^{(2),t} := \langle e_0^{(2),t} = (t, 1, 1, 1, 2) \rangle, \quad W_4^{(2),\tau} := \langle f_0^{(2),\tau} = (2, 1, 1, 1, \tau) \rangle.$$

$V_4^{(2),t} = W_4^{(3),\tau} \Leftrightarrow t = \tau = 2.$

In the case $\Lambda_3^{(3)} = \mathrm{diag}(1, \alpha_1, \alpha_2, 1, \alpha_1)$ we have

$$e_0^{(3),t} = (t, \sqrt{3}e(0), 2e(1), \sqrt{3}e(2), 0), \quad f_0^{(3),\tau} = (0, \sqrt{3}e(0), 2e(1), \sqrt{3}e(2), \tau),$$

hence

$$V_4^{(3),t} := \langle e_0^{(3),t} \rangle = W_4^{(3),\tau} := \langle f_0^{(3),\tau} \rangle \Leftrightarrow t = \tau = 0.$$

Define in the case $\Lambda_4^{(4)} = \mathrm{diag}(1, i, -1-i, 1)$

$$V_4^{(4),t} := \langle e_0^{(4),t}, e_1^{(4)}, e_2^{(4)} \rangle, \quad W_4^{(4),\tau} := \langle f_0^{(4),\tau}, f_1^{(4)}, f_2^{(4)} \rangle.$$

If we set $e(k) := \exp(2\pi ik/8)$ we get

$$\begin{array}{rcccccl}
e_0^{(4),t} = (& t, & e(0), & \sqrt{2}e(1), & 2e(2), & 2\sqrt{2}e(3) &), \\
e_1^{(4)} = (& 2\sqrt{2}, & e(1), & 0, & 0, & 0, &), \\
e_2^{(4)} = (& 3\sqrt{2}e(-1), & 3e(0), & \sqrt{2}e(1), & 0, & 0, &), \\
e_3^{(4)} = (& 2\sqrt{2}e(0), & 3e(0), & 2\sqrt{2}e(2), & 2e(3), & 0 &), \\
f_0^{(4),\tau} = (& \sqrt{2}e(-1), & e(0), & \frac{1}{\sqrt{2}}e(1), & \frac{1}{2}e(2), & \tau &), \\
f_1^{(4)} = (& 0, & 0, & 0, & e(0), & 2\sqrt{2}e(1) &), \\
f_2^{(4)} = (& 0, & 0, & \sqrt{2}e(1), & 3e(2), & 3\sqrt{2}e(3) &), \\
f_3^{(4)} = (& 0, & 2e(0), & 2\sqrt{2}e(1), & 3e(2), & 2\sqrt{2}e(3) &).
\end{array}$$

When $V_4^{(4),t} = W_4^{(4),\tau}$?

Problem, $n = 5$**.** To find eigenvectors $e_{0,5}^{(s),t_0}$, $e_{1,k}^{(s),t_1}$, $e_{2,5}^{(s),t_2}$, $e_{3,5}^{(s),t_3}$, for $\sigma_1^\Lambda(1,5)$ and $f_{0,5}^{(s),\tau_0}$, $f_{1,5}^{(s),\tau_1}$, $f_{2,5}^{(s),\tau_2}$, $f_{3,5}^{(s),\tau_3}$ for $\sigma_2^\Lambda(1,5)$ corresponding to eigenvalues $\lambda_{k,5}^{(s)} = \alpha_k^{(s)}$ in four different cases $\Lambda_5^{(s)} = \mathrm{diag}(\alpha_k^{(s)})_{k=0}^5$, $s = 2, 3, 4, 5$.

$$\Lambda_5^{(2)} = \mathrm{diag}(1, -1, 1-1, 1, 1), \quad V_5^{(2),t} := \langle e_0^{(2),t} \rangle = W_5^{(2),\tau} := \langle f_0^{(2),\tau} \rangle,$$
$$\Lambda_5^{(3)} = \mathrm{diag}(1, \alpha_1, \alpha_2, 1, \alpha_1, \alpha_2), \quad V_5^{(3),t} := \langle e_0^{(3),t} \rangle = W_5^{(3),\tau} := \langle f_0^{(3),\tau} \rangle,$$
$$\Lambda_5^{(4)} = \mathrm{diag}(1, i, -1-i, 1, i), \quad V_5^{(4),t} := \langle e_0^{(4),t} \rangle = W_5^{(4),\tau} := \langle f_0^{(4),\tau} \rangle,$$
$$\Lambda_5^{(5)} = \lambda_0 \mathrm{diag}(\alpha_k^{(5)})_{k=0}^5,$$
$$V_5^{(5),t} := \langle e_0^{(5),t}, e_1^{(5)}, e_2^{(5)}, e_3^{(5)} \rangle, \quad W_5^{(5),\tau} := \langle f_0^{(5),\tau}, f_1^{(5)}, f_2^{(5)}, f_3^{(5)} \rangle.$$

Problem, n**.** To find the eigenvectors $(e_k^{(s),t_k})_{k=0}^{n-2}$ for $\sigma_1^\Lambda(1,n)$ and $(f_k^{(s),\tau_k})_{k=0}^{n-2}$ for $\sigma_2^\Lambda(1,n)$ corresponding to eigenvalues $\lambda_{k,n}^{(s)} = \alpha_k^{(s)}$, $0 \le k \le n-2$ in $n-2$ different cases $\Lambda_n^{(s)} = \lambda_0 \mathrm{diag}(\alpha_k^{(s)})_{k=0}^n$, $2 \le s \le n$.

Define for $2 \le s \le n-3$ the subspaces

$$V_n^{(s),t} := \langle e_0^{(s),t} \rangle, \quad W_n^{(s),\tau} := \langle f_0^{(s),\tau} \rangle,$$

and

$$V_n^{(n),t} := \langle e_0^{(n),t},\, e_k^{(n)} \mid 1 \le k \le n-2 \rangle, \quad W_n^{(n),\tau} := \langle f_0^{(n),\tau},\ f_k^{(n)} \mid 1 \le k \le n-2 \rangle.$$

When $V_n^{(s),t} = W_n^{(s),\tau}$ for $0 \le s \le n-2$? $\square$

Some general formulas. Let $\Lambda_n(\alpha) = \mathrm{diag}(\alpha^{n-k})_{k=0}^n$ for some $\alpha \in \mathbb{C}$. We find the eigenvectors $e_0(\alpha)$ (resp. $f_0(\alpha)$) of the operator $\sigma_1(1,n)\Lambda_n(\alpha)$ (resp. $\Lambda_n^\sharp(\alpha)\sigma_1(2,n)$) corresponding to the eigenvalue $\lambda = 1$. We would like to find the vectors $e_0(\alpha)$ (resp. $f_0(\alpha)$) in the following form $e = (\mu^{n-k})_{k=0}^n$. We have for $n=4$

$$\begin{pmatrix} 1&4&6&4&1\\ 0&1&3&3&1\\ 0&0&1&2&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix} \Lambda_4(\alpha) \begin{pmatrix} \mu^4\\ \mu^3\\ \mu^2\\ \mu^1\\ \mu^0 \end{pmatrix} = \begin{pmatrix} \mu^4\\ \mu^3\\ \mu^2\\ \mu^1\\ \mu^0 \end{pmatrix}, \text{ or } \begin{matrix} (\alpha\mu+1)^4=\mu^4\\ (\alpha\mu+1)^3=\mu^3\\ (\alpha\mu+1)^2=\mu^2\\ (\alpha\mu+1)=\mu\\ (\alpha\mu+1)^0=\mu^0 \end{matrix}$$

so we have $\left(\frac{\alpha\mu+1}{\mu}\right)^k = 1$ for $k = 0, ..., 4$ or $\alpha\mu + 1 = \mu$. Finally $\mu = (1-\alpha)^{-1}$ and we get for $n = 4$ and for general n

$$e_0(\alpha) = ((1-\alpha)^{-(n-k)})_{k=0}^n.$$

For $f_0(\alpha)$ we get

$$\Lambda_4^\sharp(\alpha) \begin{pmatrix} 1&0&0&0&0\\ -1&1&0&0&0\\ 1&-2&1&0&0\\ -1&3&-3&1&0\\ 1&-4&6&-4&1 \end{pmatrix} \begin{pmatrix} \mu^4\\ \mu^3\\ \mu^2\\ \mu^1\\ \mu^0 \end{pmatrix} = \begin{pmatrix} \mu^4\\ \mu^3\\ \mu^2\\ \mu^1\\ \mu^0 \end{pmatrix}, \text{ or } \begin{matrix} \alpha^0\mu^4=\mu^4\\ \alpha(-\mu^4+\mu^3)=\mu^3\\ \alpha^2(\mu^4-2\mu^3+\mu^2)=\mu^2\\ \alpha^3(-\mu^4+3\mu^3+3\mu^2+\mu)=\mu\\ \alpha^4(\mu^4-4\mu^3+6\mu^2-4\mu+\mu^0)=\mu^0 \end{matrix}, \quad \begin{matrix} \alpha^0\mu^4=\mu^4\\ \alpha\mu^3(1-\mu)=\mu^3\\ \alpha^2\mu^2(1-\mu)^2=\mu^2\\ \alpha^3\mu(1-\mu)^3=\mu\\ \alpha^4(1-\mu)^4=\mu^0 \end{matrix}$$

or $\alpha^k(1-\mu)^k = 1,\ 0 \le k \le n,\ \mu = 1-\alpha^{-1}$. Finally we get

$$f_0(\alpha) = ((1-\alpha^{-1})^{n-k})_{k=0}^n.$$

For $n = 2$, $\Lambda_2 = (1,-1,1)$ and $e_2 = (2,1,2)$ we have $\sigma_1^{\Lambda_2} e_2 = \sigma_2^{\Lambda_2} e_2 = e_2$. Indeed (see (68))

$$\sigma_1^{\Lambda_2} e_2 = \begin{pmatrix} 1&-2&1\\ 0&-1&1\\ 0&0&1 \end{pmatrix} \begin{pmatrix} 2\\1\\2 \end{pmatrix} = \begin{pmatrix} 2\\1\\2 \end{pmatrix}, \quad \sigma_2^{\Lambda_2} e_2 = \begin{pmatrix} 1&0&0\\ 1&-1&0\\ 1&-2&1 \end{pmatrix} \begin{pmatrix} 2\\1\\2 \end{pmatrix} = \begin{pmatrix} 2\\1\\2 \end{pmatrix}. \tag{97}$$

For $n = 4$, $\Lambda_4 = (1,-1,1,-1,1)$ and $e_4 = (2,1,1,1,2)$ we have $\sigma_1^{\Lambda_4} e_4 = \sigma_2^{\Lambda_4} e_4 = e_4$. Indeed

$$\sigma_1^{\Lambda_n} e_4 = \begin{pmatrix} 1&-4&6&-4&1\\ 0&-1&3&-3&1\\ 0&0&1&-2&1\\ 0&0&0&-1&1\\ 0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 2\\1\\1\\1\\2 \end{pmatrix} = \begin{pmatrix} 2\\1\\1\\1\\2 \end{pmatrix}, \quad \sigma_2^{\Lambda_4} e_4 = \begin{pmatrix} 1&0&0&0&0\\ 1&-1&0&0&0\\ 1&-2&1&0&0\\ 1&-3&3&-1&0\\ 1&-4&6&-4&1 \end{pmatrix} \begin{pmatrix} 2\\1\\1\\1\\2 \end{pmatrix} = \begin{pmatrix} 2\\1\\1\\1\\2 \end{pmatrix}. \tag{98}$$

In the general case for $n=2m,\ \Lambda_n=\mathrm{diag}((-1)^k)_{k=0}^n$ and $e_n=(2,1,1,...,1,2)$ we have $\sigma_1^{\Lambda_n}e_n=\sigma_2^{\Lambda_n}e_n=e_n$. Indeed we have (see (42)) if $k\neq 0$ and $k\neq n$

$$(\sigma_1^{\Lambda_n}e_n)_k=\sum_{m=k}^{n-r}\sigma_1(1,n)_{km}(e_n)_m=\sum_{m=k}^{n-r}C_{n-k}^{n-m}(-1)^m+C_{n-k}^0=1,$$

$$(\sigma_1^{\Lambda_n}e_n)_0=\sum_{m=k}^{n}\sigma_1(1,n)_{0m}(e_n)_m=\sum_{m=0}^{n}C_n^{n-m}(-1)^m+C_n^n+C_n^0=2,$$

$$(\sigma_1^{\Lambda_n}e_n)_n=C_0^0 2=2.$$

Since $\sigma_2^{\Lambda_n}=(\sigma_1^{\Lambda_n})^\sharp$ and e_n is symmetric i.e. $(e_n)_k=(e_n)_{n-k}$ we also conclude that $\sigma_2^{\Lambda_n}e_n=e_n$.

Reducibility. We use the following notations

$$\Lambda_n^{(2)}=\mathrm{diag}(\alpha_k^{(2)})_{k=0}^n,\quad \alpha_k^{(2)}=\exp(2\pi ik/2),\quad \sigma_1^{\Lambda_n^{(2)}}=\sigma_1(1,n)\Lambda_n^{(2)},$$

$$\sigma_2^{\Lambda_n^{(2)}}=(\Lambda_n^{(2)})^\sharp\sigma_2(1,n),\ \ \mathbf{1}=(1,1,...,1),\ \ \delta_0=(1,0,...,0),\ \ \delta_n=(0,...,0,1),$$
$$e_n^{(2)}=\mathbf{1}+\delta_0+\delta_n,\ \text{for } n=2m,\ \text{and } e_n^{(2)}=\mathbf{1}-\delta_0-\delta_n \text{ for } n=2m+1.$$

Lemma 28 *For any $n\geq 2$ holds*

$$\sigma_1^{\Lambda_n^{(2)}}e_n^{(2)}=\sigma_2^{\Lambda_n^{(2)}}e_n^{(2)}=e_n^{(2)}. \tag{99}$$

PROOF. It is sufficient to prove that for $n=2m+1$ operator $\sigma_1^{\Lambda_n^{(2)}}$ (resp. $\sigma_2^{\Lambda_n^{(2)}}$) acts as follows (the vectors in the second line are the images of the corresponding vectors in the first line for example $\sigma_2^{\Lambda_n^{(2)}}\delta_0=-\mathbf{1}$):

$$\begin{pmatrix}\mathbf{1}&\delta_0&\delta_n\\-\delta_n&\delta_0&-\mathbf{1}\end{pmatrix},\ \begin{pmatrix}\mathbf{1}&\delta_0&\delta_n\\-\delta_0&-\mathbf{1}&\delta_0\end{pmatrix}, \tag{100}$$

and for $n=2m$ as follows

$$\begin{pmatrix}\mathbf{1}&\delta_0&\delta_n\\\delta_n&\delta_0&\mathbf{1}\end{pmatrix},\ \begin{pmatrix}\mathbf{1}&\delta_0&\delta_n\\\delta_0&\mathbf{1}&\delta_n\end{pmatrix}. \tag{101}$$

Indeed, in this case we get for $n=2m+1$

$$\sigma_1^{\Lambda_n^{(2)}}e_n^{(2)}=\sigma_1^{\Lambda_n^{(2)}}(\mathbf{1}-\delta_0-\delta_n)=(-\delta_n-\delta_0+\mathbf{1})=e_n^{(2)},$$

$$\sigma_2^{\Lambda_n^{(2)}}e_n^{(2)}=\sigma_2^{\Lambda_n^{(2)}}(\mathbf{1}-\delta_0-\delta_n)=(-\delta_0+\mathbf{1}-\delta_n)=e_n^{(2)}.$$

For $n=2m$ we get

$$\sigma_1^{\Lambda_n^{(2)}}e_n^{(2)}=\sigma_1^{\Lambda_n^{(2)}}(\mathbf{1}+\delta_0+\delta_n)=(\delta_n+\delta_0+\mathbf{1})=e_n^{(2)}$$

$$\sigma_2^{\Lambda_n^{(2)}} e_n^{(2)} = \sigma_2^{\Lambda_n^{(2)}} (\mathbf{1} + \delta_0 + \delta_n) = (\delta_0 + \mathbf{1} + \delta_n) = e_n^{(2)}.$$

The proof of (100) and (101) is based on the identity $\sum_{r=0}^{k}(-1)^r C_k^r = 0$. $\square$

9.4 *Counterexamples*

By Theorem 3 and 4 we conclude that for $n = 1$ two matrices

$$\sigma_1^{\Lambda}(1,1) = \begin{pmatrix} \lambda_0 & \lambda_1 \\ 0 & \lambda_1 \end{pmatrix} \text{ and } \sigma_2^{\Lambda}(1,1) = \begin{pmatrix} \lambda_1 & 0 \\ -\lambda_0 & \lambda_0 \end{pmatrix}$$

with $\lambda_1/\lambda_0 = \alpha,\ \alpha^2 - \alpha + 1 = 0$ are operator irreducible but they are not **subspace irreducible**.

For $q \neq 1,\ \Lambda_2 = I,\ n = 2$ two matrices $\sigma_1^D(q,2)$ and $\sigma_1^D(q,2)$ for $q = -1$

$$\sigma_1^D(q,2) = \begin{pmatrix} q & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \sigma_2^D(q,2) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -(1+q) & q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

are **operator irreducible** since the minors

$$M_{12}^{01}(0_2) = \begin{vmatrix} 1+q & 1 \\ 1-q & 1 \end{vmatrix} = 2q, \quad M_2^2(0_2) = 1 - q$$

can not be zero simultaneously (see section 9.1, case 3), proof of the Lemma 20). By Lemma 21 they are not **subspace irreducible** since $(2)_q = 1 + q = 0$ etc.

9.5 *Equivalence*

Theorem 5 *If two represenations $\sigma^{\Lambda}(q,n)$ and $\sigma^{\Lambda'}(q',n)$ are equivalent i.e.*

$$\sigma_i^{\Lambda}(q,n)C = C\sigma_i^{\Lambda'}(q',n),\ i = 1,2$$

for some $C \in \mathrm{GL}(n+1,\mathbb{C})$ then $q/q' = 1$ for $n = 2m$ and $(q/q')^2 = 1$ for $n = 2m-1$.

PROOF. To obtain a criteria of the equivalence it is necessary to study four cases as in the proof of the Theorem 3 (see Section 9.1) separately.

To prove the theorem it is sufficient to consider the commutation relation for some $C \in \mathrm{GL}(n+1,\mathbb{C})$

$$\lambda_0\lambda_n S(q)\Lambda_n C = C\lambda'_0\lambda'_n S(q')\Lambda'_n.$$

$\square$

Proof of the Remark 6.3 (the equivalence of the representations). In this section we index row and columns of the matrix $A \in \mathrm{Mat}(n,\mathbb{C})$ starting from 1: $A = (a_{km})_{1\leq k,m\leq n}$. It is easy to see that **for** $n=2$ the equivalence

$$\Lambda^{-1}\sigma_1^{\lambda}\Lambda = \sigma_1^{\Lambda}, \text{ and } \Lambda^{-1}\sigma_2^{\lambda}\Lambda = \sigma_2^{\Lambda} \tag{102}$$

holds. Indeed, we have

$$\sigma_1^{\lambda} = \left(\begin{smallmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{smallmatrix}\right) = \Lambda\sigma_1(1),\ \sigma_1^{\Lambda} = \sigma_1(1)\Lambda = \left(\begin{smallmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_2 \end{smallmatrix}\right), \text{ where } \sigma_1(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \Lambda = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$$

and

$$\sigma_2^{\lambda} = \left(\begin{smallmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{smallmatrix}\right) = \sigma_2(1)\Lambda^{\sharp},\ \sigma_2^{\Lambda} = \Lambda^{\sharp}\sigma_2(1) = \left(\begin{smallmatrix} \lambda_2 & 0 \\ -\lambda_1 & \lambda_1 \end{smallmatrix}\right), \text{ where } \sigma_2(1) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right).$$

Hence

$$\Lambda^{-1}\sigma_1^{\lambda}\Lambda = \sigma_1^{\Lambda} \text{ and } \Lambda^{\sharp}\sigma_2^{\lambda}(\Lambda^{\sharp})^{-1} = \sigma_2^{\Lambda}.$$

But $\Lambda^{\sharp} = \Lambda^{-1}\mathrm{det}\Lambda$, which yields (102).

We shall show that **for** $n=3$ and $C = \mathrm{diag}(1,1,\lambda_3/\lambda_2)$ the equivalence

$$\sigma_1^{\lambda} = C\sigma_1^{\Lambda}C^{-1}, \text{ and } \sigma_2^{\lambda} = C\sigma_2^{\Lambda}C^{-1} \tag{103}$$

holds. Indeed, we have

$$\sigma_1^{\Lambda} = \sigma_1(q)\Lambda \quad \text{and} \quad \sigma_2^{\Lambda} = \Lambda^{\sharp}(\sigma_1^{-1}(q^{-1}))^{\sharp},$$

$$\text{where} \quad q = \frac{\lambda_1\lambda_3}{\lambda_2^2}, \quad \sigma_1(q) = \left(\begin{smallmatrix} 1 & 1+q & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad \Lambda = \left(\begin{smallmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{smallmatrix}\right).$$

To find $\sigma_1^{-1}(q)$, $\sigma_2(q)$ and $\sigma_2^{-1}(q)$ we use the following formulas. Let X be an upper triangular matrix of infinite order with units on the principal diagonal $X = I + x = I + \sum_{k<n} x_{kn}E_{kn}$, where E_{kn} are matrix units of infinite order. Let us denote by x_{kn}^{-1} the matrix element of the inverse matrix X^{-1}

$$X^{-1} = (I+x)^{-1} = I + \sum_{k<n} x_{kn}^{-1}E_{kn}.$$

Since $XX^{-1} = X^{-1}X = I$ we have

$$\sum_{r=k}^{n} x_{kr}^{-1}x_{rn} = \sum_{r=k}^{n} x_{kr}x_{rn}^{-1} = \delta_{kn}. \tag{104}$$

The following explicit formula for x_{kn}^{-1} holds (see [16] formula (4.4))

$$x_{kk+1}^{-1} = -x_{kk+1},$$

$$x_{kn}^{-1}=-x_{kn}+\sum_{r=1}^{n-k-1}(-1)^{r+1}\sum_{k\leq i_1<i_2<...<i_r\leq n}x_{ki_1}x_{i_1i_2}...x_{i_rn},\quad k<n-1. \quad (105)$$

We have

$$\sigma_1^{-1}(q)=\begin{pmatrix}1&-(1+q)&q\\0&1&-1\\0&0&1\end{pmatrix},$$

hence

$$\sigma_2(q)=(\sigma_1^{-1}(q^{-1}))^\sharp=\begin{pmatrix}1&0&0\\-1&1&0\\q^{-1}&-(1+q^{-1})&1\end{pmatrix},\ \sigma_1^\Lambda=\sigma_1(q)\Lambda=\begin{pmatrix}\lambda_1&\lambda_1\lambda_3\lambda_2^{-1}+\lambda_2&\lambda_3\\0&\lambda_2&\lambda_3\\0&0&\lambda_3\end{pmatrix},$$

$$\sigma_2^\Lambda=\Lambda^\sharp(\sigma_1^{-1}(q^{-1}))^\sharp=\begin{pmatrix}\lambda_3&0&0\\-\lambda_2&\lambda_2&0\\\lambda_2^2\lambda_3^{-1}&-\lambda_1-\lambda_2^2\lambda_3^{-1}&\lambda_1\end{pmatrix}.$$

We compare σ_1^Λ and σ_2^Λ with the following expressions (see (37))

$$\sigma_1^\lambda=\begin{pmatrix}\lambda_1&\lambda_1\lambda_3\lambda_2^{-1}+\lambda_2&\lambda_2\\0&\lambda_2&\lambda_2\\0&0&\lambda_3\end{pmatrix},\quad \sigma_2^\lambda=\begin{pmatrix}\lambda_3&0&0\\-\lambda_2&\lambda_2&0\\\lambda_2&-\lambda_1\lambda_3\lambda_2^{-1}-\lambda_2&\lambda_1\end{pmatrix}.$$

We have

$$\sigma_1^\lambda\Lambda^{-1}=\begin{pmatrix}1&1+q&\frac{\lambda_2}{\lambda_3}\\0&1&\frac{\lambda_2}{\lambda_3}\\0&0&1\end{pmatrix}=:\sigma_1'(q)\text{ and }\sigma_1^\Lambda\Lambda^{-1}=\begin{pmatrix}1&1+q&1\\0&1&1\\0&0&1\end{pmatrix}=\sigma_1(q).$$

We see that $\sigma_1'(q)=C\sigma_1(q)C^{-1}$ where $C=\mathrm{diag}(1,1,\lambda_3/\lambda_2)$, hence

$$\sigma_1^\lambda\Lambda^{-1}=C\sigma_1^\Lambda\Lambda^{-1}C^{-1},\quad\text{so}\quad\sigma_1^\lambda=C\sigma_1^\Lambda\Lambda^{-1}C^{-1}\Lambda=C\sigma_1^\Lambda C^{-1}$$

since $\Lambda^{-1}C^{-1}\Lambda=C^{-1}$ (both C and Λ are diagonal). Further

$$(\Lambda^\sharp)^{-1}\sigma_1^\Lambda=(\sigma_1^{-1}(q^{-1}))^\sharp=\begin{pmatrix}1&0&0\\-1&1&0\\q^{-1}&-(1+q^{-1})&1\end{pmatrix},$$

and

$$(\Lambda^\sharp)^{-1}\sigma_1^\lambda=\begin{pmatrix}1&0&0\\-1&1&0\\\frac{\lambda_2}{\lambda_1}&-\frac{\lambda_3}{\lambda_2}-\frac{\lambda_2}{\lambda_1}&1\end{pmatrix}=C\begin{pmatrix}1&0&0\\-1&1&0\\q^{-1}&-(1+q^{-1})&1\end{pmatrix}C^{-1}.$$

Thus (103) holds.

For $n=4$ we get if we put $q=\left(\frac{\lambda_1\lambda_4}{\lambda_2\lambda_3}\right)^{1/2}=D^{-1}$

$$\sigma_1^\lambda=\sigma_1^\Lambda=\sigma_1(q)\Lambda\quad\text{and}\quad\sigma_2^\lambda=\sigma_2^\Lambda=\Lambda^\sharp(\sigma_1^{-1}(q^{-1}))^\sharp.$$

Indeed, we have

$$\sigma_1(q)=\begin{pmatrix}1&1+q+q^2&1+q+q^2&1\\0&1&1+q&1\\0&0&1&1\\0&0&0&1\end{pmatrix},\quad\Lambda=\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{pmatrix},$$

hence $\sigma_1^\Lambda=\sigma_1(q)\Lambda=\sigma_1^\lambda$ (see (38) and (39)). To find $\sigma_1^{-1}(q)$ we have by (104)

$$x_{kn}+\sum_{r=k+1}^{n-1}x^{-1}_{kr}x_{rn}+x^{-1}_{kn}=0,\ \ x^{-1}_{kn}+\sum_{r=k+1}^{n-1}x_{kr}x^{-1}_{rn}+x_{kn}=0,$$

$$x_{kk+1}=-x_{kk+1},\quad x^{-1}_{12}=-(1+q+q^2),\, x^{-1}_{23}=-(1+q),\quad x^{-1}_{34}=-1,$$
$$x^{-1}_{24}=-x_{24}-x^{-1}_{23}x_{34}=-1+(1+q)=q,$$
$$x^{-1}_{13}=-x_{13}-x^{-1}_{12}x_{23}=-(1+q+q^2)+(1+q)(1+q+q^2)=q(1+q+q^2),$$
$$x^{-1}_{14}=-x_{14}-x^{-1}_{12}x_{23}-x^{-1}_{13}x_{34}=-1+(1+q+q^2)(1+q)-q(1+q+q^2)=-q^3,$$
(where we have used the notation $x_{km}:=\sigma_1(q)_{km}$), hence

$$\sigma_1^{-1}(q)=\begin{pmatrix}1&-(1+q+q^2)&q(1+q+q^2)&-q^3\\0&1&-(1+q)&q\\0&0&1&-1\\0&0&0&1\end{pmatrix},\ \ \sigma_1^{-1}(q^{-1})=\begin{pmatrix}1&-(1+q^{-1}+q^{-2})&q^{-1}(1+q^{-1}+q^{-2})&-q^{-3}\\0&1&-(1+q^{-1})&q^{-1}\\0&0&1&-1\\0&0&0&1\end{pmatrix},$$

$$\sigma_2(q)=(\sigma_1^{-1}(q^{-1}))^\sharp=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\q^{-1}&-(1+q^{-1})&1&0\\-q^{-3}&q^{-1}(1+q^{-1}+q^{-2})&-(1+q^{-1}+q^{-2})&1\end{pmatrix}$$
and $\sigma_2^\Lambda=\Lambda^\sharp(\sigma_1^{-1}(q^{-1}))^\sharp=\sigma_2^\lambda$ (see (39)).

For $n=5$ if we put (see (1) and (4)) $q^{-3}=\frac{\lambda_2\lambda_4}{\lambda_1\lambda_5},\quad q^{-4}=\frac{\lambda_3^2}{\lambda_1\lambda_5}$ we get

$$\sigma_1^\Lambda=\sigma_1(q)\Lambda\quad\text{and}\quad\sigma_2^\Lambda=\Lambda^\sharp(\sigma_1^{-1}(q^{-1}))^\sharp,$$

where

$$\sigma_1(q)=\begin{pmatrix}1&(1+q)(1+q^2)&(1+q^2)(1+q+q^2)&(1+q)(1+q^2)&1\\0&1&1+q+q^2&1+q+q^2&1\\0&0&1&1+q&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\quad\Lambda=\begin{pmatrix}\lambda_1&0&0&0&0\\0&\lambda_2&0&0&0\\0&0&\lambda_3&0&0\\0&0&0&\lambda_4&0\\0&0&0&0&\lambda_5\end{pmatrix},$$

$$\sigma_1(q)^{-1}=\begin{pmatrix}1&-(1+q)(1+q^2)&q(1+q)(1+q+q^2)&-q^3(1+q)(1+q^2)&q^6\\0&1&-(1+q+q^2)&q(1+q+q^2)&-q^3\\0&0&1&-(1+q)&q\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}.$$
Setting $\gamma=(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{1/5}$ in (40)) we have

$$\sigma_1\mapsto\sigma_1^\lambda=\begin{pmatrix}\lambda_1&(1+\frac{\gamma^2}{\lambda_2\lambda_4})(\lambda_2+\frac{\gamma^3}{\lambda_3\lambda_4})&(\frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma)(1+\frac{\lambda_1\lambda_5}{\gamma^2})&(1+\frac{\lambda_2\lambda_4}{\gamma^2})(\lambda_3+\frac{\gamma^3}{\lambda_2\lambda_4})&\frac{\gamma^3}{\lambda_1\lambda_5}\\0&\lambda_2&\frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma&\frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3+\gamma&\frac{\gamma^3}{\lambda_1\lambda_5}\\0&0&\lambda_3&\frac{\gamma^3}{\lambda_1\lambda_5}+\lambda_3&\frac{\gamma^3}{\lambda_1\lambda_5}\\0&0&0&\lambda_4&\lambda_4\\0&0&0&0&\lambda_5\end{pmatrix}=$$

$$=\begin{pmatrix}1&(1+\frac{\gamma^2}{\lambda_2\lambda_4})(1+\frac{\gamma^3}{\lambda_2\lambda_3\lambda_4})&\left(1+\frac{\gamma}{\lambda_3}+\left(\frac{\gamma}{\lambda_3}\right)^2\right)(1+\frac{\lambda_1\lambda_5}{\gamma^2})&\frac{\lambda_3}{\lambda_4}(1+\frac{\lambda_2\lambda_4}{\gamma^2})(1+\frac{\gamma^3}{\lambda_2\lambda_3\lambda_4})&\frac{\gamma^3}{\lambda_1\lambda_5^2}\\0&1&1+\frac{\gamma}{\lambda_3}+\left(\frac{\gamma}{\lambda_3}\right)^2&\frac{\lambda_3}{\lambda_4}\left(1+\frac{\gamma}{\lambda_3}+\frac{\gamma^3}{\lambda_1\lambda_3\lambda_5}\right)&\frac{\gamma^3}{\lambda_1\lambda_5^2}\\0&0&1&\frac{\lambda_3}{\lambda_4}\left(1+\frac{\gamma^3}{\lambda_1\lambda_3\lambda_5}\right)&\frac{\gamma^3}{\lambda_1\lambda_5^2}\\0&0&0&1&\frac{\lambda_4}{\lambda_5}\\0&0&0&0&1\end{pmatrix}\Lambda.$$

We show the equivalence of our representation with the Tuba-Wenzl representation by finding some inversible matrix $C \in \mathrm{Mat}(5,\mathbb{C})$ such that

$$\sigma_1^{\lambda} = C^{-1}\sigma_1^{\Lambda}C.$$

Indeed, using (1) and (4) we get

$$\Lambda(q) = \mathrm{diag}(1, \frac{\lambda_2\lambda_4}{\lambda_1\lambda_5}, \frac{\lambda_3^2}{\lambda_1\lambda_5}, \frac{\lambda_2\lambda_4}{\lambda_1\lambda_5}, 1) = \mathrm{diag}(1, q^{-3}, q^{-4}, q^{-3}, 1),$$

we get $q^{-3} = \frac{\lambda_2\lambda_4}{\lambda_1\lambda_5}$, $\quad q^{-4} = \frac{\lambda_3^2}{\lambda_1\lambda_5}$ (hence $q^{-1} = \frac{\lambda_3^2}{\lambda_2\lambda_4}$). Recalling that $\gamma = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{1/5}$ we conclude that

$$\frac{\gamma^2}{\lambda_2\lambda_4} = \left(\frac{\lambda_1\lambda_5}{\lambda_2\lambda_4}\right)^{2/5}\left(\frac{\lambda_3^2}{\lambda_2\lambda_4}\right)^{1/5} = q^{6/5}q^{-1/5} = q, \quad \frac{\gamma^3}{\lambda_1\lambda_3\lambda_5} = \frac{\lambda_2\lambda_4}{\gamma^2} = q^{-1},$$

$$\frac{\gamma^3}{\lambda_2\lambda_3\lambda_4} = \left(\frac{\lambda_1\lambda_5}{\lambda_2\lambda_4}\right)^{3/5}\left(\frac{\lambda_2\lambda_4}{\lambda_3^2}\right)^{1/5} = q^{9/5}q^{1/5} = q^2, \quad \frac{\lambda_1\lambda_5}{\gamma^2} = \frac{\gamma^3}{\lambda_2\lambda_3\lambda_4} = q^2,$$

$$\frac{\gamma}{\lambda_3} = \left(\frac{\lambda_1\lambda_5}{\lambda_2\lambda_4}\right)^{1/5}\left(\frac{\lambda_2\lambda_4}{\lambda_3^2}\right)^{2/5} = q^{3/5}q^{2/5} = q, \quad \frac{\gamma^3}{\lambda_1\lambda_5^2} = \frac{\gamma^3}{\lambda_1\lambda_3\lambda_5}\frac{\lambda_3}{\lambda_5} = q^{-1}\frac{\lambda_3}{\lambda_5},$$

hence

$$\sigma_1^{\lambda} = \begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & \frac{\lambda_3}{\lambda_4}(1+q^{-1})(1+q^2) & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 1 & 1+q+q^2 & \frac{\lambda_3}{\lambda_4}(1+q+q^{-1}) & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 0 & 1 & \frac{\lambda_3}{\lambda_4}(1+q^{-1}) & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 0 & 0 & 1 & \frac{\lambda_4}{\lambda_5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\Lambda$$

$$= C_4^{-1}(q)\begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 0 & 1 & 1+q & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 0 & 0 & 1 & q^{-1}\frac{\lambda_3}{\lambda_5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\Lambda C_4(q) =$$

$$C_5^{-1}(q)C_4^{-1}(q)\begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1 & 1+q+q^2 & 1+q+q^2 & 1 \\ 0 & 0 & 1 & 1+q & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\Lambda C_4(q)C_5(q) = C^{-1}\sigma_1^{\Lambda}C,$$

where $C = C_4(q)C_5(q)$ and

$$C_4(q) = \mathrm{diag}(1,1,1,q^{-1}\frac{\lambda_3}{\lambda_4},1), \;\; C_5(q) = \mathrm{diag}(1,1,1,1,q^{-1}\frac{\lambda_3}{\lambda_5}).$$

Finally we have

$$\sigma_1^{\lambda} = C^{-1}\sigma_1^{\Lambda}C, \quad \text{where} \quad C = \mathrm{diag}(1,1,1,q^{-1}\frac{\lambda_3}{\lambda_4}, q^{-1}\frac{\lambda_3}{\lambda_5})$$

and hence σ_1^{λ} should be as follows $\sigma_1^{\lambda} = C^{-1}\sigma_2^{\Lambda}C$. $\quad\square$

11 Representations of B_3 and $q-$Pascal triangle

Proof of Theorem 1. Let us first consider the case $n = 1$. We have

$$\sigma_1^\Lambda = \sigma_1(1)\Lambda = \begin{pmatrix} \lambda_0 & \lambda_1 \\ 0 & \lambda_1 \end{pmatrix}, \text{ where } \sigma_1(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

$$\sigma_2^\Lambda = \Lambda^\sharp\sigma_2(1) = \begin{pmatrix} \lambda_1 & 0 \\ -\lambda_0 & \lambda_0 \end{pmatrix}, \text{ where } \sigma_2(1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ \Lambda^\sharp = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_0 \end{pmatrix},$$

hence

$$\sigma_1^\Lambda\sigma_2^\Lambda = \lambda_0\lambda_1 \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \lambda_0\lambda_1\sigma_1(1)\sigma_2(1),$$

$$\sigma_2^\Lambda\sigma_1^\Lambda = \begin{pmatrix} \lambda_0\lambda_1 & \lambda_1^2 \\ -\lambda_0^2 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \Lambda^\sharp\sigma_1(1)\sigma_2(1)\Lambda,$$

$$\sigma_1^\Lambda\sigma_2^\Lambda\sigma_1^\Lambda = \sigma_2^\Lambda\sigma_1^\Lambda\sigma_2^\Lambda = \lambda_0\lambda_1 \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_0 & 0 \end{pmatrix} = \lambda_0\lambda_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \lambda_0\lambda_1 S\Lambda.$$

This is (17) for $n = 1$. Let us show that (17) holds for general $n \in \mathbb{N}$.

We first show that (17): $\sigma_1^\Lambda\sigma_2^\Lambda\sigma_1^\Lambda = \sigma_2^\Lambda\sigma_1^\Lambda\sigma_2^\Lambda = \lambda_0\lambda_n S(q)\Lambda$, is equivalent with

$$\sigma_1(q)\Lambda(q)\sigma_2(q) = S(q)\sigma_1^{-1}(q), \quad \sigma_1(q)\Lambda(q)\sigma_2(q) = \sigma_2^{-1}(q)S(q). \tag{106}$$

In fact (17) is equivalent with

$$\sigma_1^\Lambda\sigma_2^\Lambda = \lambda_0\lambda_n S(q)\Lambda(\sigma_1^\Lambda)^{-1} \text{ and } \sigma_2^\Lambda\sigma_1^\Lambda = \lambda_0\lambda_n S(q)\Lambda(\sigma_2^\Lambda)^{-1}.$$

We have

$$\sigma_1^\Lambda\sigma_2^\Lambda = \lambda_0\lambda_n S(q)\Lambda(\sigma_1^\Lambda)^{-1} = \lambda_0\lambda_n S(q)\sigma_1^{-1}(q), \tag{107}$$

$$\begin{aligned}\sigma_2^\Lambda\sigma_1^\Lambda = \lambda_0\lambda_n S(q)\Lambda(\sigma_2^\Lambda)^{-1} = \lambda_0\lambda_n S(q)\Lambda\sigma_2(q)^{-1}(\Lambda^\sharp)^{-1} = \\ \Lambda^\sharp S(q)\sigma_2(q)^{-1}\lambda_0\lambda_n(\Lambda\Lambda^\sharp)^{-1}\Lambda = \Lambda^\sharp S(q)\sigma_2(q)^{-1}\Lambda(q)^{-1}\Lambda,\end{aligned} \tag{108}$$

(where we used the relation $\Lambda\Lambda^\sharp = \lambda_0\lambda_n\Lambda(q)$, see (4) and $S(q)\Lambda = \Lambda^\sharp S(q)$). On the other hand we get

$$\sigma_1^\Lambda\sigma_2^\Lambda = \sigma_1(q)\Lambda\Lambda^\sharp\sigma_2(q) = \lambda_0\lambda_n\sigma_1(q)\Lambda(q)\sigma_2(q). \tag{109}$$

Comparing (107) with (109) we conclude that the first equality in (17) and the first part of (106) are equivalent. Further we have

$$\sigma_2^\Lambda\sigma_1^\Lambda = \Lambda^\sharp\sigma_2(q)\sigma_1(q)\Lambda.$$

Comparing (108) with the latter equation we conclude that the second equality in (17) and the second part of (106) are equivalent.

To prove (106) **for general** $n \in \mathbb{N}$, we give in Lemma 29 the explicit formulas for $\sigma_1^{-1}(q)$, $\sigma_2(q)$ and $\sigma_2^{-1}(q)$ (compare with Lemma 4.1, Section 3). Let us recall also the notation (see(1))

$$q_n = q^{\frac{(n-1)n}{2}}, \ n \in \mathbb{N}.$$

Lemma 29 *Let the operator* $\sigma_1(q) = (\sigma_1(q)_{km})_{0\leq k,m\leq n}$ *be defined by* $\sigma_1(q)_{km} = C_{n-k}^{n-m}(q)$*. Then for the operators* $\sigma_1^{-1}(q)$*,* $\sigma_2(q)$ *and* $\sigma_2^{-1}(q)$ *we have respectively*

$$\sigma_1(q)_{km} = C_{n-k}^{n-m}(q), \quad \sigma_1^{-1}(q)_{km} = (-1)^{k+m} q_{m-k} C_{n-k}^{n-m}(q) \tag{110}$$

and

$$\sigma_2(q)_{km} = (-1)^{k+m} q_{k-m}^{-1} C_k^m(q^{-1}), \quad \sigma_2^{-1}(q)_{km} = C_k^m(q^{-1}). \tag{111}$$

PROOF. The equality $\sigma_1^{-1}(q)_{km} = (-1)^{k+m} q_{m-k} C_{n-k}^{n-m}(q)$ is equivalent with

$$\sum_{r=k}^{n} \sigma_1(q)_{kr} \sigma_1^{-1}(q)_{rm} = \sum_{r=k}^{n} C_{n-k}^{n-r}(q)(-1)^{r+m} q_{m-r} C_{n-r}^{n-m}(q) = \delta_{km}, \tag{112}$$

and the equality $\sigma_2^{-1}(q)_{km} = C_k^m(q^{-1})$ is equivalent with the following

$$\sum_{r=k}^{n} \sigma_2(q)_{kr} \sigma_2^{-1}(q)_{rm} = \sum_{r=k}^{n} (-1)^{k+r} q_{k-r}^{-1} C_k^r(q^{-1}) C_r^m(q^{-1}) = \delta_{km}. \tag{113}$$

The identities (112) and (113) hold however by (120) (see the proof in Section 8). $\square$

We shall prove now (106) for general $n \in \mathbb{N}$:

$$\sigma_1(q)\Lambda(q)\sigma_2(q) = S(q)\sigma_1^{-1}(q),$$

$$\text{i.e } \sum_{r=0}^{n} \sigma_1(q)_{kr} \Lambda(q)_{rr} \sigma_2(q)_{rm} = S(q)_{k,n-k} \sigma_1^{-1}(q)_{n-k,m} \tag{114}$$

and

$$\sigma_1(q)\Lambda(q)\sigma_2(q) = \sigma_2^{-1}(q)S(q),$$

$$\text{i.e } \sum_{r=k}^{n} \sigma_1(q)_{kr} \Lambda(q)_{rr} \sigma_2(q)_{rm} = \sigma_2^{-1}(q)_{k,n-m} S(q)_{n-m,m}. \tag{115}$$

Using (1) and (16) we have

$$S(q) = (S(q)_{km})_{0\leq k,m\leq n}, \quad S(q)_{km} = q_k^{-1}(-1)^k \delta_{k+m,n},$$

$$\Lambda(q) = \operatorname{diag}(q_{rn})_{r=0}^{n}, \text{ where } q_{rn}^{-1} := \frac{q_n}{q_r q_{n-r}}.$$

Then by (110), (111) we get

$$\sum_{r=0}^{n}\sigma_1(q)_{kr}\Lambda(q)_{rr}\sigma_2(q)_{rm}=\sum_{r=0}^{n}C_{n-k}^{n-r}(q)\frac{q_rq_{n-r}}{q_n}(-1)^{r+m}q_{r-m}^{-1}C_r^m(q^{-1}),\quad(116)$$

$$(S(q)\sigma_1^{-1}(q))_{km}=S(q)_{k,n-k}\sigma_1^{-1}(q)_{n-k,m}=q_k^{-1}(-1)^k(-1)^{n-k+m}q_{m-n+k}C_k^{n-m}(q)$$
$$=(-1)^{n+m}\frac{q_{m-n+k}}{q_k}C_k^{n-m}(q),\quad(117)$$

$$(\sigma_2^{-1}(q)S(q))_{km}=\sigma_2^{-1}(q)_{k,n-m}S(q)_{n-m,m}=C_k^{n-m}(q^{-1})q_{n-m}^{-1}(-1)^{n-m}.\quad(118)$$

Using (116), (117) and (121) we get (114). To prove that $(S(q)\sigma_1^{-1}(q))_{km}=(\sigma_2^{-1}(q)S(q))_{km}$ it is sufficient to show that

$$C_n^k(q)=q_{kn}^{-1}C_n^k(q^{-1})=\frac{q_n}{q_kq_{n-k}}C_n^k(q^{-1})\quad 0\leq k\leq n.\quad(119)$$

Indeed if (119) holds we have

$$C_k^{n-m}(q)=q_{k,n-m}^{-1}C_k^{n-m}(q^{-1})=\frac{q_k}{q_{m-n+k}q_{n-m}}C_k^{n-m}(q^{-1}).$$

Comparing (117) and (118) we conclude that $(S(q)\sigma_1^{-1}(q))_{km}=(\sigma_2^{-1}(q)S(q))_{km}$.

To prove (119) it is sufficient to use the definition (8) of $C_n^k(q)$

$$C_n^k(q)=\frac{(1-q)(1-q^2)...(1-q^n)}{(1-q)(1-q^2)...(1-q^k)(1-q)(1-q^2)...(1-q^{n-k})},$$

$$C_n^k(q^{-1})=\frac{(1-q^{-1})(1-q^{-2})...(1-q^{-n})}{(1-q^{-1})(1-q^{-2})...(1-q^{-k})(1-q^{-1})(1-q^{-2})...(1-q^{-(n-k)})},$$

recall that $q_k=q^{\frac{(k-1)k}{2}}=q^{1+2+...+k-1}$ and observe that $\frac{q_{n+1}}{q_{k+1}q_{n+1-k}}=\frac{q_n}{q_kq_{n-k}}$. Hence (114) and (115) are both proven, which implies (106). $\square$

Remark 30 *We illustrate the identity (106) and Lemma 29 for $n=2,3,4$.*

For $n=2$ we have by (3), (18) and (16)

$$\sigma_1(q)=\begin{pmatrix}1&1+q&1\\0&1&1\\0&0&1\end{pmatrix},\quad\sigma_1^{-1}(q)=\begin{pmatrix}1&-(1+q)&q\\0&1&-1\\0&0&1\end{pmatrix},\quad S(q)=\begin{pmatrix}0&0&1\\0&-1&0\\q^{-1}&0&0\end{pmatrix},$$

$$\sigma_2(q)=\begin{pmatrix}1&0&0\\-1&1&0\\q^{-1}&-(1+q^{-1})&1\end{pmatrix},\quad\sigma_2^{-1}(q)=\begin{pmatrix}1&0&0\\1&1&0\\1&(1+q^{-1})&1\end{pmatrix},\quad\Lambda(q)=\begin{pmatrix}1&0&0\\0&q^{-1}&0\\0&0&1\end{pmatrix}.$$

Finally we get

$$\sigma_1(q)\Lambda(q)\sigma_2(q)=\begin{pmatrix}1&1+q&1\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&q^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\q^{-1}&-(1+q^{-1})&1\end{pmatrix}=\begin{pmatrix}0&0&1\\0&-1&1\\q^{-1}&-(1+q^{-1})&1\end{pmatrix},$$

$$S(q)\sigma_1^{-1}(q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ q^{-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -(1+q) & q \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ q^{-1} & -(1+q^{-1}) & 1 \end{pmatrix},$$

and

$$\sigma_2(q)^{-1}S(q) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & (1+q^{-1}) & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ q^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ q^{-1} & -(1+q^{-1}) & 1 \end{pmatrix}.$$

This is (106) for $n = 2$.

For $n = 3$ we have by (3), (18) and (16)

$$\sigma_1(q) = \begin{pmatrix} 1 & (1+q+q^2) & (1+q+q^2) & 1 \\ 0 & 1 & (1+q) & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^{-1}(q) = \begin{pmatrix} 1 & -(1+q+q^2) & q(1+q+q^2) & -q^3 \\ 0 & 1 & -(1+q) & q \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_1^{-1}(q^{-1}) = \begin{pmatrix} 1 & -(1+q^{-1}+q^{-2}) & q^{-1}(1+q^{-1}+q^{-2}) & -q^{-3} \\ 0 & 1 & -(1+q^{-1}) & q^{-1} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S(q) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & q^{-1} & 0 & 0 \\ q^{-3} & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_2(q) = (\sigma_1^{-1}(q^{-1}))^\sharp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ q^{-1} & -(1+q^{-1}) & 1 & 0 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix},$$

$$\sigma_2^{-1}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & (1+q^{-1}) & 1 & 0 \\ 1 & (1+q^{-1}+q^{-2}) & (1+q^{-1}+q^{-2}) & 1 \end{pmatrix}, \quad \Lambda(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We verify that (106) holds, moreover that

$$\sigma_1(q)\Lambda(q)\sigma_2(q) = S(q)\sigma_1^{-1}(q) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & q^{-1} & -(1+q^{-1}) & 1 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix}.$$

Indeed we have

$$\sigma_1(q)\Lambda(q)\sigma_2(q) = \begin{pmatrix} 1 & (1+q+q^2) & (1+q+q^2) & 1 \\ 0 & 1 & (1+q) & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ q^{-1} & -(1+q^{-1}) & 1 & 0 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & q^{-1} & -(1+q^{-1}) & 1 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix}.$$

$$S(q)\sigma_1^{-1}(q) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & q^{-1} & 0 & 0 \\ -q^{-3} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -(1+q+q^2) & q(1+q+q^2) & -q^3 \\ 0 & 1 & -(1+q) & q \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & q^{-1} & -(1+q^{-1}) & 1 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix} = \sigma_1(q)\Lambda(q)\sigma_2(q),$$

and

$$\sigma_2(q)^{-1}S(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & (1+q^{-1}) & 1 & 0 \\ 1 & (1+q^{-1}+q^{-2}) & (1+q^{-1}+q^{-2}) & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & q^{-1} & 0 & 0 \\ -q^{-3} & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & q^{-1} & -(1+q^{-1}) & 1 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 \end{pmatrix} = \sigma_1(q)\Lambda(q)\sigma_2(q).$$

This is (106) for n=3. **For** $n = 4$ we have

$$\sigma_1(q) = \begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q+q^2)(1+q^2) & (1+q)(1+q^2) & 1 \\ 0 & 1 & (1+q+q^2) & (1+q+q^2) & 1 \\ 0 & 0 & 1 & (1+q) & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda(q) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q^{-3} & 0 & 0 & 0 \\ 0 & 0 & q^{-4} & 0 & 0 \\ 0 & 0 & 0 & q^{-3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1^{-1}(q^{-1}) = \begin{pmatrix} 1 & -(1+q^{-1})(1+q^{-2}) & q^{-1}(1+q^{-1}+q^{-2})(1+q^{-2}) & -q^{-3}(1+q^{-1})(1+q^{-2}) & q^{-6} \\ 0 & 1 & -(1+q^{-1}+q^{-2}) & q^{-1}(1+q^{-1}+q^{-2}) & -q^{-3} \\ 0 & 0 & 1 & -(1+q^{-1}) & q^{-1} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\sigma_2(q) = (\sigma_1^{-1}(q^{-1}))^\sharp =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ q^{-1} & -(1+q^{-1}) & 1 & 0 & 0 \\ -q^{-3} & q^{-1}(1+q^{-1}+q^{-2}) & -(1+q^{-1}+q^{-2}) & 1 & 0 \\ q^{-6} & -q^{-3}(1+q^{-1})(1+q^{-2}) & q^{-1}(1+q^{-1}+q^{-2})(1+q^{-2}) & -(1+q^{-1})(1+q^{-2}) & 1 \end{pmatrix},$$

$$\sigma_2^{-1}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & (1+q^{-1}) & 1 & 0 & 0 \\ 1 & (1+q^{-1}+q^{-2}) & (1+q^{-1}+q^{-2}) & 1 & 0 \\ 1 & (1+q^{-1})(1+q^{-2}) & (1+q^{-1}+q^{-2})(1+q^{-2}) & (1+q^{-1})(1+q^{-2}) & 1 \end{pmatrix},$$

hence $\sigma_1(q)\Lambda(q)\sigma_2(q) = S(q)\sigma_1^{-1}(q) = \sigma_2^{-1}(q)S(q)$.

12 Combinatorial identities for $q-$binomial coefficients

Lemma 31 *The following identities hold*

$$\sum_{i=0}^{n} C_m^i(q)(-1)^{i+j}q_{i-j}C_i^j(q) = \sum_{i=0}^{n}(-1)^{i+m}q_{m-i}C_m^i(q)C_i^j(q) = \delta_{mj}, \qquad (120)$$

$$\sum_{r=0}^{n} C_{n-k}^{n-r}(q)\frac{q_r q_{n-r}}{q_n}(-1)^{n-r}q_{r-m}^{-1}C_r^m(q^{-1}) = \frac{q_{k-(n-m)}}{q_k}C_k^{n-m}(q). \qquad (121)$$

Remark 32 *For* $q = 1$ *(120) and (121) reduce to the well known identities (122) and (123) (see [20, p.4] and [20, p.8 eq. (5)]):*

$$\sum_{i=0}^{n}(-1)^{i+m}\binom{m}{i}\binom{i}{j} = \sum_{i=0}^{n}(-1)^{i+j}\binom{m}{i}\binom{i}{j} = \delta_{mj}, \qquad (122)$$

$$\sum_{i=0}^{n}(-1)^{i}\binom{m}{i}\binom{n-i}{n-j} = \sum_{i=0}^{n}(-1)^{i}\binom{m}{i}\binom{n-i}{j-i} = \binom{n-m}{j}. \qquad (123)$$

PROOF. We prove the identities by induction. For $n = 0$ we have in both cases $1 = 1$. Let (120) holds for $n \in \mathbb{N}$. We prove that this holds then for $n+1$

i.e.

$$\sum_{i=0}^{n+1} C_m^i(q)(-1)^{i+j} q_{i-j} C_i^j(q) = \delta_{mj} \quad 0 \le m, j \le n+1.$$

For $0 \le m, j \le n$ this hold by the assumption. It is sufficient to consider $m = n+1$. We have by (13)

$$\sum_{i=0}^{n+1} C_{n+1}^i(q)(-1)^{i+j} q_{i-j} C_i^j(q) = \sum_{i=0}^{n+1}(-1)^{i+j} q_{i-j} \left(C_n^{i-1}(q) + q^i C_n^i(q)\right) C_i^j(q) =$$

$$\sum_{i=0}^{n+1}(-1)^{i+j} q_{i-j} C_n^{i-1}(q) C_i^j(q) + \sum_{i=0}^{n+1} C_n^i(q)(-1)^{i+j} q^i q_{i-j} C_i^j(q) =$$

$$\sum_{i=0}^{n+1}(-1)^{i+j} q_{i-j} C_n^{i-1}(q) \left(C_{i-1}^{j-1}(q) + q^j C_{i-1}^j(q)\right) + \sum_{i=0}^{n+1} C_n^i(q)(-1)^{i+j} q^i q_{i-j} C_i^j(q) =$$

$$\sum_{i=0}^{n+1}(-1)^{i+j} q_{i-j} C_n^{i-1}(q) C_{i-1}^{j-1}(q) +$$

$$\sum_{i=0}^{n+1}(-1)^{i+j} q_{i-j} C_n^{i-1}(q) q^j C_{i-1}^j(q) + \sum_{i=0}^{n+1} C_n^i(q)(-1)^{i+j} q^i q_{i-j} C_i^j(q) = \delta_{n,j-1} = \delta_{n+1,j}$$

since the sum of the last two terms gives 0 by $q^j q_{i+1-j} = q^i q_{i-j}$. The latter relation follows from $q_{n+1} = q^n q_n$.

Let us suppose that (121) is true for $n \in \mathbb{N}$. We prove that then this holds for $n+1$ i.e.

$$\sum_{r=0}^{n+1}(-1)^{n+1-r} C_{n+1-k}^{n+1-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} C_r^m(q^{-1}) = \frac{q_{k-(n+1-m)}}{q_k} C_k^{n+1-m}(q).$$

Indeed, by (13) the left hand side of the latter equation is equal to

$$\sum_{r=0}^{n+1}(a_r + b_r)(c_r + d_r) = \sum_{r=0}^{n+1}[a_r(c_r + d_r) + b_r c_r + b_r d_r] :=$$

$$\sum_{r=0}^{n+1}(-1)^{n+1-r} \left(C_{n-k}^{n-r}(q) + q^{n+1-r} C_{n-k}^{n+1-r}(q)\right) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} \times$$

$$\left(C_{r-1}^{m-1}(q^{-1}) + q^{-m} C_{r-1}^m(q^{-1})\right) = \sum_{r=0}^{n+1}(-1)^{n+1-r} C_{n-k}^{n-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} C_r^m(q^{-1})$$

$$+ \sum_{r=0}^{n+1}(-1)^{n+1-r} q^{n+1-r} C_{n-k}^{n+1-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} C_{r-1}^{m-1}(q^{-1})$$

$$+ \sum_{r=0}^{n+1}(-1)^{n+1-r} q^{n+1-r} C_{n-k}^{n+1-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} C_{r-1}^m(q^{-1}).$$

Since $q_r = q_{r-1} q^{r-1}$ and $q_{n+1} = q_n q^n$ we have

$$q^{n+1-r} \frac{q_r q_{n+1-r}}{q_{n+1}} = \frac{q_{r-1} q_{n-(r-1)}}{q_n} = \frac{q_s q_{n-s}}{q_n}. \tag{124}$$

Setting $s = r - 1$ we get by the assumption of the induction (121)

$$\sum_{r=0}^{n+1} b_r c_r = \sum_{s=0}^{n} (-1)^{n-s} C_{n-k}^{n-s}(q) \frac{q_s q_{n-s}}{q_n} q_{s-(m-1)}^{-1} C_s^{m-1}(q^{-1})$$

$$= \frac{q_{k-[n-(m-1)]}}{q_k} C_k^{n-(m-1)}(q) = \frac{q_{k-(n+1-m)}}{q_k} C_k^{n+1-m}(q).$$

We prove that $\sum_{r=0}^{n+1} [a_r(c_r + d_r) + b_r d_r] = 0$. Indeed $\sum_{r=0}^{n+1} a_r(c_r + d_r) =$

$$\sum_{r=0}^{n+1} (-1)^{n+1-r} C_{n-k}^{n-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} C_r^m(q^{-1}) =$$

$$-\sum_{r=0}^{n} (-1)^{n-r} C_{n-k}^{n-r}(q) \frac{q_r q_{n-r}}{q_n} q^{-r} q_{r-m}^{-1} C_r^m(q^{-1}).$$

If we set $s = r - 1$, use (124) and $q_{s+1-m} q^m = q_{s-m} q^{s-m} q^m = q_{s-m} q^s$ we get

$$\sum_{r=0}^{n+1} b_r d_r = \sum_{r=0}^{n+1} (-1)^{n+1-r} q^{n+1-r} C_{n-k}^{n+1-r}(q) \frac{q_r q_{n+1-r}}{q_{n+1}} q_{r-m}^{-1} q^{-m} C_{r-1}^m(q^{-1}) =$$

$$\sum_{r=0}^{n} (-1)^{n-s} C_{n-k}^{n-s}(q) \frac{q_s q_{n-s}}{q_n} q_{s+1-m}^{-1} q^{-m} C_{r-1}^m(q^{-1})$$

hence $\sum_{r=0}^{n+1} [a_r(c_r + d_r) + b_r d_r] = 0$ and (121) is proven for general $n \in \mathbb{N}$. $\square$

13 A $q-$analogue of the results of E. Ferrand

Denote by $\Phi(q) = \Phi_n(q)$ the endomorphism of the space $\mathbb{C}^n[X]$ of polynomials of degree n with complex coefficients, which maps a polynomial $p(x)$ to the polynomial $p_q(X + 1)$, where for $p(X) = \sum_{k=0}^n a_k X^k$ we define

$$p(X) \overset{\Phi_n(q)}{\mapsto} p_q(1 + X) := \sum_{k=0}^{n} a_k (1 + X)_q^k.$$

Denote by $\Psi(q) = \Psi_n(q)$ the endomorphism of $\mathbb{C}^n[X]$ which maps a polynomial $p(X)$ to the following polynomial

$$p(X) \overset{\Psi_n(q)}{\mapsto} \sum_{k=0}^{n} a_k q_k (1 - X)_{q^{-1}}^{n-k} X^k,$$

compare with the expression for Ψ (see Section 3)

$$p(X) \overset{\Psi}{\mapsto} (1 - X)^n p\left(\frac{X}{1 - X}\right) = \sum_{k=0}^{n} a_k (1 - X)^{n-k} X^k.$$

Theorem 33 *The endomorphisms $\Phi(q)$ and $\Psi(q)$ satisfy braid-like relation $\Phi(q)\Psi(q)\Phi(q)=\Psi(q)\Phi(q)\Psi(q)$.*

PROOF. We have by (14)

$$X^k \overset{\Phi(q)}{\mapsto} (1+X)_q^k=(1+X)(1+qX)...(1+q^{k-1}X)=\sum_{r=0}^{k} q^{r(r-1)/2}C_k^r(q)x^r,$$

hence $\Phi_{rk}(q)=q^{r(r-1)/2}C_k^r(q)=q_rC_k^r(q)$ and by (3) and (19) we conclude that

$$\Phi(q)=\Phi_n(q)=D_n(q)\sigma_1^s(q)=(\sigma_1(q)D_n^\sharp(q))^s. \tag{125}$$

Indeed $\sigma_1(q)_{km}=C_{n-k}^{n-m}(q)$ hence $\sigma_1^s(q)_{km}=C_m^k(q)$ (we recall that $a_{ij}^s=a_{n-j,n-i}$). For the operator $\Psi_n(q)$ we get

$$X^k \overset{\Psi(q)}{\mapsto} q_{n-k}(1-X)_{q^{-1}}^{n-k}X^k \tag{126}$$

$$=q_{n-k}\sum_{s=0}^{n-k}(-1)^s q_s^{-1}C_{n-k}^s(q^{-1})X^sX^k=q_{n-k}\sum_{r=k}^{n}(-1)^{r+k}q_{r-k}^{-1}C_{n-k}^{r-k}(q^{-1})X^r,$$

hence $\Psi_{rk}(q)=(-1)^{r+k}q_{n-k}q_{r-k}^{-1}C_{n-k}^{r-k}(q^{-1})$ and by (18) and (19) we conclude that

$$\Psi(q)=\Psi_n(q)=\sigma_2^s(q)D_n^s(q)=(D_n(q)\sigma_2(q))^s. \tag{127}$$

Indeed $\sigma_2(q)_{rk}=(-1)^{r+k}q_{r-k}^{-1}C_r^k(q^{-1})$, hence $\sigma_2^s(q)_{rk}=(-1)^{r+k}q_{r-k}^{-1}C_{n-k}^{n-r}(q^{-1})$, and $D_n^s(q)=\mathrm{diag}(q_{n-r})_{r=0}^n$. We note that

$$(\Phi^{-1}(q)p)(X)=p_{q^{-1}}(X-1). \tag{128}$$

To finish the proof we use Remark 4.4, Section 3, representation (21). $\square$

In the particular cases for $n=2$ and $n=3$ we get

$$\Phi_2(q)=D_2(q)\sigma_1^s(q)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&q\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&1+q\\0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&1+q\\0&0&q\end{pmatrix}, \tag{129}$$

$$\Phi_3(q)=D_3(q)\sigma_1^s(q)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&q&0\\0&0&0&q^3\end{pmatrix}\begin{pmatrix}1&1&1&1\\0&1&1+q&1+q+q^2\\0&0&1&1+q+q^2\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1\\0&1&1+q&1+q+q^2\\0&0&q&q(1+q+q^2)\\0&0&0&q^3\end{pmatrix}, \tag{130}$$

$$\Psi_2(q)=\sigma_2^s(q)D_2^s(q)=\begin{pmatrix}1&0&0\\-(1+q^{-1})&1&0\\q^{-1}&-1&1\end{pmatrix}\begin{pmatrix}q&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}q&0&0\\-(1+q)&1&0\\1&-1&1\end{pmatrix}, \tag{131}$$

$$\Psi_3(q)=\begin{pmatrix}1&0&0&0\\-(1+q^{-1}+q^{-2})&1&0&0\\q^{-1}(1+q^{-1}+q^{-2})&-(1+q^{-1})&1&0\\-q^{-3}&q^{-1}&-1&1\end{pmatrix}\begin{pmatrix}q^3&0&0&0\\0&q&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}q^3&0&0&0\\-q(1+q+q^2)&q&0&0\\(1+q+q^2)&-(1+q)&1&0\\-1&1&-1&1\end{pmatrix}. \tag{132}$$

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